Differentiable Reflection Quasigroups, Bruck Loops, and Gyrogroups

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A useful variant of the notion of an affine connection applicable to both finite and infinite dimensional manifolds is the notion of a spray. K.-H. Neeb has shown that any smooth symmetric space in the sense of Loos admits a canonical associated spray. We show that all sprays give rise to a locally defined midpoint operation, which in the symmetric space setting is locally the left quotient of the symmetry or reflection operation. The geodesics (with respect to the spray) are then precisely the continous maps from open intervals of the reals that are locally midpoint preserving.

If one specializes to those smooth symmetric spaces that are reflection quasigroups, then a standard reversible construction associates to the quasigroup its "geodesic loop," a uniquely divisible differentiable Bruck loop, or equivalently, a uniquely divisible gyrocommutative gyrogroup. We show that the exponential mapping allows one to induce an operation of "scalar multiplication" so that the gyrogroup exhibits many analogies to a vector space. There is also a close relationship between the gyroaddition and parallel translation with respect to the canonical spray. In general, there are grounds for viewing the smooth gyrogroups that arise in this setting as loop analogs of real vector spaces. Important examples to which the theory applies include the core of positive elements of a C^*-algebra, analytic hyperbolic geometry, Möbius geometry, and the qubit loop.