On the Baer correspondance of finite loops and groups

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A *quasigroup* is a set Q endowed with three binary operations $x \cdot y$, x/y, $x\backslash y$ such that the relations

$$x \cdot (x\backslash y) = (y/x) \cdot x = y$$

hold. A *loop* is a quasigroup with unit element e: $e \cdot x = x \cdot e = x$. Although loops are often referred to as "non-associative groups", they have a much weaker inner structure; I even dear to say that in general, they have no inner structure at all. General finite loops and quasigroups are interesting object in combinatorics under the name *Latin squares.*

The systematic study of loops and quasigroups started around 1930, mainly with a motivation from projective and differential geometry. Since the beginning, many attemps has been made to translate loop theoretical problems to group theory. The main tools have been the *left* and *right multiplication maps* $L_a : x \mapsto ax$, $R_a : x \mapsto ax$, and, the (left, right and full) multiplication group generated by them.

In this talk, we concentrate on two equationally defined classes of loops:

Right Bol loops: $((xy)z)y = x((yz)y)$,
Moufang loops: $((xy)z)y = x(y(zy))$.

The category of Moufang loops corresponds to the category of groups with triality. We say that $(G, \{., \sigma, \rho\})$ is a group with triality if $(G, \cdot)$ is a group, $\sigma, \rho \in \mathrm{Aut}(G)$, $\sigma^2 = (\sigma\rho)^2 = \rho^3 = \mathrm{id}$ and for all $g \in G$, the *triality identity*

$$[\sigma, g]\, [\sigma, g]^{\rho}\, [\sigma, g]^{\rho^2} = 1$$

holds. The fact that non-isomorphic groups with triality can correspond to isomorphic Moufang loops makes this functorial equivalence complicated.

The situation of right Bol loops is similar. There, the corresponding group theoretical structure is called a *Bol loop folder.* The quintuples (G, H, K, α, β) is a Bol loop folder if G is a group, H is a subgroup and K is a subset of G such that $1 \in K$, and for all $x, y \in K$, $xyx \in K$. Moreover the maps $\alpha : G \to H$, $\beta : G \to K$ must satisfy

$$g = \alpha(g)\beta(g)$$

for all $g \in G$. Subfolders and morhisms are defined in the obvious way. Again, the problem is that a Bol loop can have non-isomorphic loop folders.

In my talk, I will explain these functors and show some examples how to use them for constructing and classifying interesting classes of Bol and Moufang loops.