Modes, modals, and barycentric algebras

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A *mode* (A, Ω) is an idempotent and entropic algebra: Each singleton subset $\{a\}$ of A is a subalgebra, and each operation $\omega : A^{\omega\tau} \to A$ (with arity $\omega\tau$) is a homomorphism. A *modal* $(A, +, \Omega)$ is a semilattice $(A, +)$ and a mode (A, Ω) such that each mode operation ω distributes over the semilattice operation:

$$a_1 \dots (a_i + a_i') \dots a_{\omega\tau}\omega = a_1 \dots a_i \dots a_{\omega\tau}\omega + a_1 \dots a_i' \dots a_{\omega\tau}\omega$$

for $1 \le i \le \omega\tau$. For a mode A, the set AS of nonempty subalgebras of A forms a modal $(AS, +, \Omega)$ under the join and complex products of subalgebras. In particular, modes enjoy the self-reproducing property that the subalgebra set AS of a mode A again forms a mode, with A as a subalgebra.

The real line carries operations of *complementation* $p' = 1 - p$, *dual multiplication* $p \circ q = (p'q')'$, and *implication*

$$p \to q = \textbf{if } (p = 0) \textbf{ then } 1 \textbf{ else } q/p\,.$$

A *barycentric algebra* (A, I°) is a set A with a binary operation $xy\,\underline{p}$ for each element p of the open real unit interval I° satisfying *idempotence* $xx\,\underline{p} = x$, *skew-commutativity* $xy\,\underline{p} = yx\,\underline{p'}$, and *skew-associativity* $xy\,\underline{p}\,z\,\underline{q} = x\,yz\,\big(\underline{p \circ q \to q}\big)\,\underline{p \circ q}$. Examples are given by semilattices with $xy\,\underline{p} = x \cup y$ and convex sets with $xy\,\underline{p} = xp' + yp$. Barycentric algebras are modes.

Both theoretical aspects and applications of these structures will be discussed in the presentation.