Rectangularity theorem for conservative algebras

Libor Barto
e-mail: libor.barto@gmail.com
Charles University in Prague, Czech Republic

The Constraint Satisfaction Problem over $\mathbb{A} = (A, \Gamma)$, where A is a finite set and Γ is a set of relations on A (sometimes assumed to be finite), is a decision problem asking whether an input positive primitive (pp-) sentence over $\mathbb{A}$ is true. A pp-sentence is a sentence formed from relations in Γ using conjunction and existential quantification, i.e. a sentence of the form

$$(\exists x_1 \in A)\ (\exists x_2 \in A) \dots R_1(variables) \wedge R_2(vars) \wedge \dots \wedge R_k(vars),$$

where $R_i \in \Gamma$. This decision problem is denoted by CSP($\mathbb{A}$). Many well known problems, such as satisfiability problems for boolean formulas (3-SAT, HORN-SAT, ...), colorings of graphs, solving systems of equations over finite domains, etc. are examples of CSPs over suitably chosen $\mathbb{A}$.

The central theoretical problem in this area is the Dichotomy Conjecture of Feder and Vardi.

Conjecture 1 (CSP Dichotomy Conjecture)**.** *For every* $\mathbb{A}$*, the problem* CSP($\mathbb{A}$) *is either poly-time, or NP-complete.*

The algebraic approach to the Dichotomy Conjecture is based on the fact that the complexity of CSP($\mathbb{A}$) is determined by the clone $\mathbf{A}$ of all operations compatible with all relations in Γ. See Pawel Idziak's talk for an overview of t he algebraic approach to CSP.

One of the biggest achievements in the area is a result of Bulatov, which confirms the conjecture in the case that $\mathbb{A}$ contains all unary relations:

Theorem 2 (Conservative CSP Dichotomy)**.** *Assume that* $\mathbb{A}$ *contains all unary relations over A. If the corresponding* $\mathbf{A}$ *is a Taylor clone, then the problem* CSP($\mathbb{A}$) *is poly-time, otherwise it is NP-complete.*

A clone $\mathbf{A}$ is said to be Taylor, if it contains a Taylor operation. This condition has a growing number of equivalent reformulations. For instance, it is known that a finite clone $\mathbf{A}$ is Taylor, iff the variety generated by $\mathbf{A}$ omits type 1, iff $\mathbf{A}$ satisfies some non-trivial idempotent Malcev condition, iff $\mathbf{A}$ contains a weak near-unanimity operation (or a Siggers operation , or a cyclic operation), ... A variety is called Taylor, if all algebras in the variety have Taylor clones of term operations.

The original proof of Theorem 2 is very technical, complicated and long (the paper has 80 pages). Using a refinement of our work with Marcin Kozik, we are able to give a natural and quite short proof.

In my talk I will present the algebraic core of this new proof – the rectangularity theorem for conservative algebras.

To introduce the rectangularity theorem, we need the concept of an absorbing subalgebra of an algebra.

Definition 3. Let $\mathbf{A}$ be an algebra (or a clone). A subalgebra $\mathbf{B}$ of $\mathbf{A}$ is called an *absorbing subalgebra* of $\mathbf{A}$, if there exists a term operation t of $\mathbf{A}$ such that for every coordinate i and for every $b_1, \ldots, b_n \in A$, such that $b_j \in B$ for all $j \neq i$, we have $t(b_1, \ldots, b_n) \in B$. We denote this fact by $\mathbf{B} \triangleleft \mathbf{A}$.

It is not hard to see that the relation $\triangleleft$ is transitive, therefore the next definition is quite natural.

Definition 4. Let $\mathbf{A}$ be an algebra. A subalgebra $\mathbf{B}$ of $\mathbf{A}$ is called a *minimal absorbing subalgebra* (notation $\mathbf{B} \triangleleft\!\triangleleft\, \mathbf{A}$), if $\mathbf{B} \triangleleft \mathbf{A}$ and $\mathbf{B}$ has no proper absorbing subalgebra.

Recall that an algebra $\mathbf{A}$ (or a clone) is called conservative, if every operation f of the algebra (clone) is conservative, that is $f(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\}$ for any $a_1, \ldots, a_n \in A$. Note that the clone of compatible operations of a relational structure containing all unary relations is conservative.

Theorem 5 (Rectangularity theorem for conservative algebras)**.** *Let $\mathbf{A}_1, \ldots, \mathbf{A}_n$ be finite conservative algebras from a Taylor variety, let $\mathbf{B}_i \triangleleft\!\triangleleft\, \mathbf{A}_i$, $i = 1, \ldots, n$, let R be a subdirect subuniverse of $\mathbf{A}_1 \times \ldots \mathbf{A}_n$, let $S = R \cap (B_1 \times \cdots \times B_n)$ and assume $S \neq \emptyset$.*

Consider the following equivalence $\sim$ on $\{1, 2, \ldots, n\}$:

$$i \sim j \text{ iff } \forall (a_1, \ldots, a_n) \in R \ (a_i \in B_i \ \&\ a_j \in B_j) \text{ or } (a_i \notin B_i \ \&\ a_j \notin B_j)$$

Let $D_1, \ldots, D_k$ be a complete list of $\sim$-blocks.

Then a tuple $\mathbf{b} = (b_1, \ldots, b_n) \in B_1 \times \cdots \times B_n$ belongs to S if and only if the restriction of $\mathbf{b}$ to D_j is in the projection of R to D_j for every $j = 1, \ldots, k$.

In particular, if all $\sim$-blocks are one element, then $S = B_1 \times \cdots \times B_n$.