Ore extensions over Weak σ-rigid rings and $\sigma(*)$-rings

Vijay Kumar Bhat
e-mail: vijaykumarbhat2000@yahoo.com
SMVD University, India

Throughout this paper R will denote an associative ring with identity $1 \neq 0$. The field of complex numbers is denoted by $\mathbb{C}$, the field of rational numbers is denoted by $\mathbb{Q}$, the ring of integers is denoted by $\mathbb{Z}$, and the set of positive integers is denoted by $\mathbb{N}$. The set of prime ideals of R is denoted by $Spec(R)$. The set of minimal prime ideals of R is denoted by $Min.Spec(R)$. The prime radical and the set of nilpotent elements of R are denoted by $P(R)$ and $N(R)$ respectively. We note that for a commutative ring $P(R)$ and $N(R)$.

Let now R be a ring and σ an endomorphism of a ring R. Recall that R is said to be a $\sigma(*)$-ring if $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$, where $P(R)$ is the prime radical of R (Kwak [6]).

Example 1: (Example 2 of Kwak [6]): Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field. Then $P(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ Let $\sigma : R \to R$ be defined by $\sigma\Big(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\Big)$ $= \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. Then it can be seen that σ is an endomorphism of R and R is a $\sigma(*)$-ring. We also recall that R is said to be a weak σ-rigid ring if $a\sigma(a) \in N(R)$ if and only if $a \in N(R)$ for $a \in R$, where $N(R)$ is the set of nilpotent elements of R (Ouyang [9]).

Example 2: (Example (2.1) of Ouyang [9]: Let σ be an endomorphism of a ring R such that R is a σ-rigid ring. Let

$$A = \Big\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid \text{a, b, c, d} \in R \Big\}$$

be a subring of $T_3(R)$, the ring of upper triangular matrices over R. Now σ can be extended to an endomorphism $\overline{\sigma}$ of A by $\overline{\sigma}((a_{ij})) = (\sigma(a_{ij}))$. The it can be seen that A is a weak $\overline{\sigma}$-rigid ring.

Recall that a ring R is 2-primal if and only if $N(R) = P(R)$ if and only if the prime radical is a completely semiprime ideal. An ideal I of a ring R is called completely semiprime if $a^2 \in I$ implies $a \in I$ for $a \in R$. We note that a reduced is 2-primal.

Ore Extensions (skew polynomial rings):

Now let R be a ring and σ an endomorphism of R and δ is a σ-derivation of R. Recall that the skew polynomial ring $R[x; \sigma, \delta]$ is the set of polynomials

$$\{\sum_{i=0}^{n} x^i a_i,\ a_i \in R,\ n \in \mathbb{N}\} \text{ where } \mathbb{N} \text{ is the set of positive integers}$$

with usual addition of polynomials and multiplication subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We take any $f(x) \in R[x;\sigma,\delta]$ to be of the form $f(x) = \sum_{i=0}^{n} x^i a_i$, $a_i \in R$ as in McConnell and Robson [8]. We denote $R[x;\sigma,\delta]$ by $O(R)$. Skew-polynomial rings have been of interest to many authors. For example [1], [2], [3], [4], [6], [7], [8], [9].

In this paper we give a relation between a $\sigma(*)$-ring and a weak σ-rigid ring. We also give a necessary and sufficient condition for a Noetherian ring to be a weak σ-rigid ring. Let σ be an endomorphism of a ring R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then σ can be extended to an endomorphism (say $\overline{\sigma}$) of $R[x;\sigma,\delta]$ and δ can be extended to a $\overline{\sigma}$-derivation (say $\overline{\delta}$) of $R[x;\sigma,\delta]$.

Before we state the results, we recall that a completely prime ideal in a ring R is any (prime) ideal such that R/P is a domain (Chapter 9 of Goodearl and Warfield [5]). By definition we note that every completely prime ideal of a ring R is a prime ideal, but the converse need not be true.

Example 3: Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} = M_2(\mathbb{Z})$. If p is a prime number, then the ideal $P = M_2(p\mathbb{Z})$ is a prime ideal of R, but is not completely prime, since for $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, we have $ab \in P$, even though $a \notin P$ and $b \notin P$.

With this we prove the following results:

(1) Let R be a Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring. Then $U \in Min.Spec(R)$ implies that $UO(R) = U[x;\sigma,\delta]$ is a completely prime ideal of $O(R) = R[x;\sigma,\delta]$.
(2) Let R be a Noetherian ring. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring. Then R is a weak σ-rigid ring. Conversely a 2-primal weak σ-rigid ring is a $\sigma(*)$-ring.
(3) Let R be a commutative Noetherian ring. Let σ be an automorphism of R. Then R is a weak σ-rigid ring implies that $N(R)$ is completely semiprime.
(4) Let R be a commutative Noetherian ring. Let σ be an automorphism of R. Then R is a 2-primal weak σ-rigid ring if and only if for each minimal prime U of R, $\sigma(U) = U$ and U is completely prime ideal of R.
(5) Let R be a commutative Noetherian ring. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring. Then $O(N(R)) = N(O(R))$.

Using the above results we prove the following main Theorem:

Theorem A: Let R be a 2-primal commutative Noetherian ring. Let σ be an automorphism of R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then R is a weak σ-rigid ring implies that $O(R) = R[x;\sigma,\delta]$ is a weak $\overline{\sigma}$-rigid ring.

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