Polynomially representable semirings[1]

Ivan Chajda⋆
e-mail: chajda@inf.upol.cz
Palacký University Olomouc, Czech Republic

Günther Eigenthaler
e-mail: g.eigenthaler@tuwien.ac.at
Vienna University of Technology, Austria

We characterize semirings which can be represented by an algebra of binary polynomials of the form $a \cdot x + y$ where the operations are compositions of functions. Furthermore, we classify which algebras with two binary and two nullary operations can be represented in this way and how these algebras are related to semirings.

In what follows, we will consider algebras $\mathcal{A} = (A; +, \cdot, 0, 1)$ of type $(2, 2, 0, 0)$. For $a \in A$ we define a binary polynomial function

$$f_a(x, y) = a \cdot x + y.$$

Denote by $F(A) = \{f_a(x, y); a \in A\}$. On the set $F(A)$ we introduce the following function compositions:

$$(f_a \oplus f_b)(x, y) = f_a(x, f_b(x, y)),$$

$$(f_a \circ f_b)(x, y) = f_a(f_b(x, y), y).$$

If we assume that $\mathcal{A} = (A; +, \cdot, 0, 1)$ satisfies the identities $1 \cdot x = x$, $0 \cdot x = 0$ and $0 + x = x$ then clearly $f_0(x, y) = y$ and $f_1(x, y) = x + y$. The algebra

$$\mathcal{F}(A) = (F(A); \oplus, \circ, y, x + y)$$

will be called the *function algebra assigned to* $\mathcal{A}$.

Definition 1. A *semiring* is an algebra $\mathcal{A} = (A; +, \cdot, 0, 1)$ such that

(i) $+$ is associative and commutative,

(ii) $\cdot$ is associative,

(iii) $+$ and $\cdot$ satisfy the left and right distributive laws

$$x \cdot (y + z) = x \cdot y + x \cdot z,$$

$$(x + y) \cdot z = x \cdot z + y \cdot z,$$

(iv) $+$ and $\cdot$ satisfy the idenities $x + 0 = x$,
$x \cdot 1 = x = 1 \cdot x$,
$x \cdot 0 = 0 = 0 \cdot x$.

A semiring $\mathcal{A}$ is called *simple* (see [8]) if it satisfies the identity

$$x + 1 = 1.$$

Theorem 1. *For a semiring* $\mathcal{A} = (A; +, \cdot, 0, 1)$*, the following conditions are equivalent:*

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(1) $\mathcal{F}(A) = (F(A); \oplus, \circ, y, x+y)$ *is a semiring and* φ *is an isomorphism of* $\mathcal{A}$ *onto* $\mathcal{F}(A)$*,*

(2) $\mathcal{A}$ *is simple.*

One can mention that commutativity of $+$ was not used in the proof of Theorem 1. This motivates us to establish for what algebras $\mathcal{A} = (A; +, \cdot, 0, 1)$ of type $(2, 2, 0, 0)$ the mapping φ is an isomorphism of $\mathcal{A}$ on the functional algebra $\mathcal{F}(A)$. We are going to show that, in such a case, $\mathcal{A}$ will share a majority of properties of a semiring except commutatively of $+$ and the right distributivity.

Theorem 2. *Let* $\mathcal{A} = (A; +, \cdot, 0, 1)$ *be an algebra of type* $(2, 2, 0, 0)$ *satisfying* $x + 0 = x = 0 + x$, $x \cdot 0 = 0$ *and* $x \cdot 1 = x$*. The following conditions are equivalent:*

(1) $\varphi : a \mapsto f_a(x, y) = a \cdot x + y$ *is an isomorphism of* $\mathcal{A}$ *onto* $\mathcal{F}(A) = (F(A); \oplus, \circ, y, x + y)$*,*

(2) $+$ *and* $\cdot$ *are associative and* $\mathcal{A}$ *satisfies*

$(x + y) \cdot z = x \cdot z + y \cdot z,$

$x \cdot (y + z) + z = x \cdot y + z.$

The last identity of (2) in Theorem 2 is rather curious. Hence, we are interested how this is related with left-distributivity and the absorption law. The complete answer is contained in the following lemma and example.

Lemma 2. *Let* $\mathcal{A} = (A; +, \cdot, 0, 1)$ *be an algebra of type* $(2, 2, 0, 0)$ *satisfying associativity of* $+$ *and* $0 + x = x$, $x \cdot 0 = 0$*. Then:*

(a) If $\mathcal{A}$ *satisfies left-distributivity* $x \cdot (y + z) = x \cdot y + x \cdot z$ *and the absorption law then* $\mathcal{A}$ *satisfies also*

$$x \cdot (y + z) + z = x \cdot y + z.$$

(b) If $\mathcal{A}$ *satisfies* $x \cdot (y + z) + z = x \cdot y + z$ *then it satisfies the absorption law.*

Due to Lemma 2, we are interested in the question if the identity $x \cdot (y + z) + z = x \cdot y + z$ together with associativity of $+$ and the identities $0 + x = x, x \cdot 0 = 0$ can infer left-distributivity. The following example shows that it is not the case.

Example. Consider a three-element set $A = \{0, a, 1\}$ and an algebra $\mathcal{A} = (A; +, \cdot, 0, 1)$ where the operations $+$ and $\cdot$ are determinded by the tables:

$+$	0	a	1
0	0	a	1
a	a	a	1
1	1	1	1

$\cdot$	0	a	1
0	0	a	0
a	0	a	a
1	0	a	1

It is an easy exercise to verify that $\mathcal{A}$ satisfies all the identities of (2) of Theorem 2 and $0 + x = x = x + 0$, $x \cdot 0 = 0$, $x \cdot 1 = x$. However, $\mathcal{A}$ does not satisfy left-distributivity since e.g.

$$0 \cdot (1 + a) = 0 \cdot 1 = 0 \neq a = 0 + a = 0 \cdot 1 + 0 \cdot a.$$