

Weak relative pseudocomplementation on semilattices

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1. Introduction. A meet semilattice is said to be *weakly relatively pseudocomplemented*, or just *wr-pseudocomplemented*, if, for every element x and every $y \leq x$, all the maxima

$$(1) \quad \max\{u: y = u \wedge x\}$$

exist. This concept goes back to [5], where the congruence lattice of a semilattice was shown to be wr-pseudocomplemented (without using the term). Weak relative pseudocomplements in congruence lattices of various structures are discussed also in several other papers; they are uncovered as well in lattices of closure operators and certain subalgebra lattices of semigroups and groups. Also, every meet-semidistributive algebraic (in particular, finite) lattice is wr-pseudocomplemented. Proposition 2.1 of [7] implies that sectionally pseudocomplemented semilattices (i.e., semilattices with pseudocomplemented principal filters) are just wr-pseudocomplemented semilattices. Moreover, a meet semilattice with the largest element is wr-pseudocomplemented if and only if every closed interval in it is pseudocomplemented.

Let S be a weakly relatively pseudocomplemented semilattice (it necessary has the largest element 1). Given elements $p, a \in S$ with $p \leq a$, we denote the weak relative pseudocomplement of a in S (i.e. the maximum (1) for $x = a$ and $y = p$) by $a * p$. Thus, the operation $*$ is partial, and a natural question arises how to extend it to a total operation in a reasonable way and when it is possible to do. Several (and different) answers to the question can be found in the literature (see below). In this paper, we study some elementary properties of wr-pseudocomplemented semilattices, show that the class \mathbf{AWR}^\wedge of all semilattices with “totalized” wr-pseudocomplementation form a variety with nice congruence properties, isolate an interesting subvariety of it, and compare the latter with other known subvarieties of \mathbf{AWR}^\wedge .

For example, every Brouwerian, i.e. relatively pseudocomplemented semilattice belongs to \mathbf{AWR}^\wedge . The following result extends to semilattices Theorem 3 of [8] proved for lattices with pseudocomplemented closed intervals.

Theorem 1. *A semilattice is relatively pseudocomplemented if and only if it is weakly relatively pseudocomplemented and modular. Hence, a modular wr-pseudocomplemented semilattice is distributive.*

2. Weak relative annihilators. The notion of a relative annihilator was introduced for lattices in [4] and for meet semilattices in [9]; it was also shown in these papers that a lattice, resp. semilattice is distributive if and only if every relative annihilator is an ideal. Also, any relative pseudocomplement is actually the maximum element of some relative annihilator and conversely. We generalise these facts

for meet semi-distributivity and weak relative pseudocomplementation in S . In particular, a wr-pseudocomplemented semilattice is meet-semidistributive in the sense that, for all x, y, z, p ,

$$\text{if } x \wedge y = p = x \wedge z, \text{ then } p = x \wedge k \text{ for some } k \geq y, z.$$

A *weak annihilator* $\langle a, b \rangle$ of a relative to b is the subset $\{x \in S : a \wedge x = a \wedge b\}$. Therefore, $a * p = \max\langle a, p \rangle$ when $p \leq a$. A semilattice in which $\max\langle a, p \rangle$ exists for all a and b is a *semi-Brouwerian semilattice* in the sense of [7].

3. Extending wr-pseudocomplementation. We call a wr-pseudocomplemented semilattice S *augmented* if it is equipped with a total binary operation, say \rightarrow , extending $*$:

$$(2) \quad x \rightarrow y = x * y \text{ whenever } y \leq x,$$

and consider such semilattices as algebras of kind $(A, \wedge, \rightarrow, 1)$. Any operation \rightarrow satisfying (2) is an *augmented wr-pseudocomplementation*. Let us denote the class of all augmented wr-semilattices by \mathbf{AWR}^\wedge .

We introduce the following useful abridgment:

$$x \rightsquigarrow y := x \rightarrow (x \wedge y).$$

Theorem 2. *The class \mathbf{AWR}^\wedge is a variety characterised by the semilattice axioms and identities*

$$x \wedge (x \rightsquigarrow y) \leq y, \quad y \leq x \rightsquigarrow y.$$

We may consider semi-Brouwerian semilattices as augmented, where $a \rightarrow b$ stands for $\max\langle a, b \rangle$. Then $x \rightarrow y = x * (x \wedge y)$. Hence, the class \mathbf{SBS} of all semi-Brouwerian semilattices is subvariety of \mathbf{AWR}^\wedge determined by the additional axiom $x \rightsquigarrow y = x \rightarrow y$. In particular, the algebra $(A, \wedge, \rightsquigarrow, 1)$ is always a semi-Brouwerian semilattice.

Theorem 3. *Every algebra $A \in \mathbf{AWR}^\wedge$ is arithmetical at 1 and weakly regular at 1 (1-regular). Hence,*

- (a) *A is congruence distributive,*
- (b) *the congruence kernels $1/\theta$ of A form a lattice $N(A)$,*
- (c) *the mapping $\theta \mapsto 1/\theta$ is an isomorphism from the congruence lattice $C(A)$ onto $N(A)$.*

4. Pseudo-Brouwerian semilattices. Every congruence kernel of an \mathbf{AWR}^\wedge -algebra is a filter of its underlying semilattice. We now are going to find out when the converse holds. Given a filter F of an algebra $S \in \mathbf{AWR}^\wedge$, we denote by F^\sharp the binary relation $\{(x, y) : a \wedge x = a \wedge y \text{ for some } a \in F\}$. F is its kernel: $1/(F^\sharp) = F$. Now, if θ is a congruence of S , then $F = 1/\theta$ if and only if $\theta = F^\sharp$.

Theorem 4. *A filter F is a congruence kernel if and only if the following holds in S for every $x \in F$ (and all y, z):*

$$(3) \quad x \wedge (y \rightarrow z) = x \wedge ((x \wedge y) \rightarrow (x \wedge z)).$$

By a *pseudo-Brouwerian semilattice* we shall mean any AWR^\wedge -algebra in which (3) is an identity. The name is motivated by the fact that all Brouwerian semilattices belong to the variety PBS of all pseudo-Brouwerian semilattices and also to the following result.

Proposition 5. *An algebra $(S, \rightarrow, 1) \in AWR^\wedge$ is a pseudo-Brouwerian semilattice if and only if $(S, \rightsquigarrow, 1)$ is a Brouwerian semilattice.*

The next two theorems are the main results of the paper. See [2] for the concepts of a congruence orderable and Fregean variety.

Theorem 6. *Let S be an AWR^\wedge -algebra, and let $F(S)$ be its lattice of filters. Then $F(S) = N(S)$ if and only if S belongs to PBS .*

Theorem 7. *The variety PBS is arithmetical and congruence orderable. Hence, it is Fregean.*

5. Some other subvarieties of AWR^\wedge . Theorem 3 of [3] states that a meet semilattice with the greatest element is sectionally pseudocomplemented if and only if it admits a (total) binary operation \rightarrow subject to axioms

$$(4) \quad x \rightarrow x = 1, \quad x \wedge (x \rightarrow y) = x \wedge y, \quad x \rightarrow (x \wedge y) = x \rightarrow y.$$

Its proof implies that the operation \rightarrow obeys also (2). Against this background, sectionally pseudocomplemented semilattices are treated in [3] as algebras of kind $(S, \wedge, \rightarrow, 1)$ satisfying the above identities. The class SPS_{HK} of all these algebras is thus a subvariety of AWR^\wedge and includes SBS as a proper subclass.

In [6], H.P. Sankappanavar defines a semi-Heyting algebra to be a bounded lattice with an additional operation \rightarrow fulfilling (3) and the first two identities in (4). He also proposes to call a semi-Brouwerian semilattice any meet semilattice with 1 and operation \rightarrow which is subject to these identities. Of course, this class SBS_H of algebras differs from SBS ; actually, it can be shown to be equal to PBS . On the other hand, SBS_H is a proper subclass of SPS_{HK} . It is proved in [6], along with other results, that a semilattice S from SBS_H has isomorphic lattices of filters and congruences and that S is subdirectly irreducible if and only if it has a unique (in fact, the greatest) coatom.

One more proper subvariety of AWR^\wedge , non-comparable with those just considered, is the class of sectionally j -pseudocomplemented semilattices mentioned in [[1], Corollary 4].

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