Sums and tolerances of lattices

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Sums and *tolerances* are important concepts in lattice theory, and perhaps, with some exceptions, only in lattice theory.

1. An easy way to sums

Given a congruence Θ of a lattice L, we can form the quotient lattice $K = L/\Theta$, and the Θ-blocks, which are lattices that form a K-indexed system $\{A \mid A \in K\}$. If we want to recreate L from the lattice K and the system $\{A \mid A \in K\}$, then we form the *sum* of a K-indexed system of lattices. Sum is also called *Płonka sum* or *Graczyńska sum*, and it is one of the main tools for studying products of lattice varieties; see, for instance, [1], [7], [8], [9], [12], [14]. For simplicity, here we consider complete lattices only.

For a K-indexed system of lattices, one has to describe how the summands are related. This is usually done by a pair of functors, one being the right adjoint of the other, see [4] for historical details. Now we offer a single functor, which is an easier to visualize . Let $L_1 = (L_1, \leq_1)$ and $L_2 = (L_2, \leq_2)$ be lattices. Roughly speaking, a relation $\rho \subseteq L_1 \times L_2$ will be called an *atop relation*, if taking disjoint copies of L_1 and L_2 and putting L_2 atop L_1 modulo ρ (that is, adding ρ to the union of $\leq_1$ and $\leq_2$), we obtain a complete lattice.

Theorem 1 ([4])**.** *The class $\mathcal{C}$ of complete lattices, as objects, together with atop relations, as morphisms, the lattice orderings, as identities, and the usual relational product, as operation, constitute a category.*

As described in [4], summable systems of lattices can be defined as a functor from K to $\mathcal{C}$, and sums can be treated accordingly.

2. Tolerances as homomorphic images of congruences

By a *tolerance* we mean a reflexive, symmetric, compatible relation; see [2] for a survey. While tolerances are congruences in congruence permutable varieties, we have four arguments for their importance in Lattice Theory. Firstly, term func tions of a finite lattice L are exactly those isotone functions that preserves all tolerances, see [13]. Secondly, tolerances are, explicitely or implicitly present in several gluings of lattices, see [6], [11], and [12], for example.

By a *block* of a tolerance $\rho \subseteq L^2$, we mean a maximal subset X of L such that $X^2 \subseteq \rho$. Blocks are convex sublattices. Note that $(x, y) \in \rho$ iff there is a block X of ρ such that $x, y \in X$. If ρ is a tolerance of an algebra A, f is an n-ary operation of A, and $X_1, \ldots, X_n$ are blocks of ρ, then Zorn's lemma yields a block Y of ρ su ch that $f(x_1, \ldots, x_n) \in Y$, for all $x_i \in X_i$. While this Y is not unique in general, the following theorem, which is the third argument, holds.

Theorem 2 ([3]). *Let ρ be a tolerance of a lattice L, and let L/ρ be the set of all blocks of ρ. Then Y above is unique; this makes L/ρ an algebra $(L/\rho, \vee, \wedge)$ in the obvious way. This $(L/\rho, \vee, \wedge)$ is a lattice.*

For an alternative approach to L/ρ see [10]. If Θ is a congruence of an algebra A and $\varphi\colon A \to B$ is a surjective homomorphism, then $\varphi(\Theta) = \{(\varphi(x), \varphi(y)) : (x, y) \in \rho\}$ is clearly a tolerance of B. Our fourth argument is

Theorem 3 ([5]). *Let ρ be a tolerance of a lattice L. Then there exist a lattice M, a congruence Θ on M, and a surjective homomorphism $\varphi\colon M \to L$ such that $\rho = \varphi(\Theta)$.*

3. A link between sums and tolerances

The first proof of Theorem 3 (for the finite case only) ran as follows: we took a sum of the blocks of ρ, which are lattices. However, in the talk we give a simpler proof.

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