Relatively congruence-modular quasivarieties and their relatively congruence-distributive sub-quasivarieties

Janusz Czelakowski
e-mail: jczel@math.uni.opole.pl
University of Opole, Poland

Let $\boldsymbol{Q}$ be a quasivariety. If $\boldsymbol{A} \in \boldsymbol{Q}$, then $\mathrm{Con}_{\boldsymbol{Q}}(\boldsymbol{A})$ is the set of $\mathbf{Q}$-congruences of $\boldsymbol{A}$:

$$\mathrm{Con}_{\boldsymbol{Q}}(\boldsymbol{A}) := \{\Phi \in \mathrm{Con}(\boldsymbol{A}) : \boldsymbol{A}/\Phi \in \boldsymbol{Q}\}.$$

$\mathrm{Con}_{\mathbf{Q}}(\boldsymbol{A})$ is an algebraic lattice. The members of $\mathrm{Con}_{\mathbf{Q}}(\boldsymbol{A})$ are called $\mathbf{Q}$-*congruences.* $\Theta^{\boldsymbol{A}}_{\mathbf{Q}}(X)$ is the least $\mathbf{Q}$-congruence in $\boldsymbol{A}$ that contains $X \subseteq A^2$.

1. The equational commutator

Let $\alpha(\underline{x}, \underline{y}, \underline{z}, \underline{w}, \underline{u}) := \alpha(x_1, ..., x_m, y_1, ..., y_m, z_1, ..., z_n, w_1, ..., w_n, u_1, ..., u_k)$ and $\beta(\underline{x}, \underline{y}, \underline{z}, \underline{w}, \underline{u}) := \beta(x_1, ..., x_m, y_1, ..., y_m, z_1, ..., z_n, w_1, ..., w_n, u_1, ..., u_k)$ be terms in Te_τ built up with at most the variables $\underline{x} = x_1, ..., x_m, \underline{y} = y_1, ..., y_m, \underline{z} = z_1, ..., z_n, \underline{w} = w_1, ..., w_n$, and possibly other variables $\underline{u} = u_1, ..., u_k$. ($\underline{x}$ and $\underline{y}$ are of the same length; similarly $\underline{z}$ and $\underline{w}$.)

Definitions 1.1. Let $\mathbf{K}$ be a class of algebras.

A. $\alpha(\underline{x}, \underline{y}, \underline{z}, \underline{w}, \underline{u}) \approx \beta(\underline{x}, \underline{y}, \underline{z}, \underline{w}, \underline{u})$ is a *commutator equation for* $\mathbf{K}$ in the variables $\underline{x}, \underline{y}$ and $\underline{z}, \underline{w}$ if the equations

$$\alpha(\underline{x}, \underline{x}, \underline{z}, \underline{w}, \underline{u}) \approx \beta(\underline{x}, \underline{x}, \underline{z}, \underline{w}, \underline{u}) \text{ and } \alpha(\underline{x}, \underline{y}, \underline{z}, \underline{z}, \underline{u}) \approx \beta((\underline{x}, \underline{y}, \underline{z}, \underline{z}, \underline{u})$$

are valid in $\mathbf{K}$.

B. A *quaternary commutator equation* for $\mathbf{K}$ (with parameters) is a commutator equation $\alpha(x, y, z, w, \underline{u}) \approx \beta(x, y, z, w, \underline{u})$ for $\mathbf{K}$ in single variables x, y and z, w. (This means that the equations $\alpha(x, x, z, w, \underline{u}) \approx \beta(x, x, z, w, \underline{u})$ and $\alpha(x, y, z, z, \underline{u}) \approx \beta(x, y, z, z, \underline{u})$ are $\mathbf{K}$-valid.) ☐

Note. It follows from the above definition that $\alpha(\underline{x}, \underline{y}, \underline{z}, \underline{w}, \underline{u}) \approx \beta(x, \underline{y}, \underline{z}, \underline{w}, \underline{u})$ is a commutator equation for $\mathbf{K}$ (in the variables $\underline{x}, \underline{y}$ and $\underline{z}, \underline{w}$) iff it is a commutator equation (in $\underline{x}, \underline{y}$ and $\underline{z}, \underline{w}$) for the variety $\boldsymbol{Va}(\mathbf{K})$ generated by $\mathbf{K}$. Consequently, the classes $\mathbf{K}$, $\boldsymbol{Qv}(\mathbf{K})$ and $\boldsymbol{Va}(\mathbf{K})$ possess the same commutator equations. ☐

If Φ is a congruence of an algebra $\boldsymbol{A}$ and $\underline{a} = \langle a_1, ..., a_m\rangle, \underline{b} = \langle b_1, ..., b_m\rangle$ are sequences of elements of A of the same length, we write $\underline{a} \equiv \underline{b}\,(\Phi)$ to indicate that $a_i \equiv b_i\,(\Phi)$ for $i = 1, ..., m$.

Definition 1.2. (Czelakowski [1]). Let $\boldsymbol{A} \in \mathbf{Q}$, where $\mathbf{Q}$ is a quasivariety, and let Φ and Ψ be $\mathbf{Q}$-congruences on $\boldsymbol{A}$. The *equationally defined commutator* of Φ and Ψ on $\boldsymbol{A}$ relative to $\mathbf{Q}$ (the equational commutator, for short), in symbols

$[\Phi, \Psi]_{\boldsymbol{A}}$,

is the least $\mathbf{Q}$-congruence on $\boldsymbol{A}$ which contains the set of pairs:

$\{\langle \alpha(\underline{a}, \underline{b}, \underline{c}, \underline{d}, \underline{e}), \beta(\underline{a}, \underline{b}, \underline{c}, \underline{d}, \underline{e})\rangle : \alpha(\underline{x}, \underline{y}, \underline{z}, \underline{w}, \underline{u}) \approx \beta(\underline{x}, \underline{y}, \underline{z}, \underline{w}, \underline{u})$
is a commutator equation for $\mathbf{Q}$, $\underline{a} \equiv \underline{b}\,(\Phi)$, $\underline{c} \equiv \underline{d}\,(\Psi)$, and $\underline{e} \in A^{<\omega}\}$. $\square$

Theorem 1.3. *Let $\mathbf{Q}$ be a quasivariety of algebras, $\boldsymbol{A} \in \mathbf{Q}$, and $\Phi, \Psi \in Con_{\mathbf{Q}}(\boldsymbol{A})$. Then:*

(i) $[\Phi, \Psi]_{\boldsymbol{A}}$ *is a $\mathbf{Q}$-congruence on $\boldsymbol{A}$;*

(ii) $[\Phi, \Psi]_{\boldsymbol{A}} \subseteq \Phi \cap \Psi$*;*

(iii) $[\Phi, \Psi]_{\boldsymbol{A}} = [\Phi, \Psi]_{\boldsymbol{A}}$*;*

(iv) *The equational commutator is monotone in both arguments, i.e., if Φ_1, Φ_2 and Ψ_1, Ψ_2 are $\mathbf{Q}$-congruences on $\boldsymbol{A}$, then $\Phi_1 \subseteq \Phi_2$ and $\Psi_1 \subseteq \Psi_2$ implies $[\Phi_1, \Psi]_{\boldsymbol{A}} \subseteq [\Phi_2, \Psi]_{\boldsymbol{A}}$ and $[\Phi, \Psi_1]_{\boldsymbol{A}} \subseteq [\Phi, \Psi_2]_{\boldsymbol{A}}$.* $\square$

2. Additivity of the equational commutator

Let $\mathbf{Q}$ be a quasivariety of algebras and $\boldsymbol{A} \in \mathbf{Q}$. The equational commutator is additive on $\boldsymbol{A}$ if for any set $\{\Psi_i : i \in I\}$ of $\mathbf{Q}$-congruences of $\boldsymbol{A}$ and any $\Psi \in \mathrm{Con}_{\mathbf{Q}}(\boldsymbol{A})$:

(C1) $[\sup_{\mathbf{Q}}\{\Psi_i : i \in I\}, \Psi]_{\boldsymbol{A}} = \sup_{\mathbf{Q}}\{[\Psi_i, \Psi]_{\boldsymbol{A}} : i \in I\}$

in the lattice $\mathrm{Con}_{\mathbf{Q}}(\boldsymbol{A})$.

Theorem 2.1. *Let $\mathbf{Q}$ be a quasivariety of algebras. The following conditions are equivalent:*

(1) *The equational commutator is additive on the algebras of $\mathbf{Q}$.*

(2) *There exists a set $\Delta_0(x, y, z, w, \underline{u})$ of quaternary commutator equations for $\mathbf{Q}$ such that for every algebra $\boldsymbol{A} \in \mathbf{Q}$ and for every pair of sets $X, Y \subseteq A^2$,*
$[\Theta^{\boldsymbol{A}}_{\mathbf{Q}}(X), \Theta \boldsymbol{A}_{\mathbf{Q}}(Y)]_{\boldsymbol{A}} =$
$\Theta^{\boldsymbol{A}}_{\mathbf{Q}}(\{\langle \alpha(a, b, c, d, \underline{e}), \beta(a, b, c, d, \underline{e})\rangle : \alpha \approx \beta \in \Delta_0, \langle a, b\rangle \in X,$
$\langle c, d\rangle \in Y, \underline{e} \in A^k\})$. $\square$

If (2) holds, $\Delta_0(x, y, z, w, \underline{u})$ is said to *generate* the equational commutator for $\mathbf{Q}$.

We need one more property of the commutator:

(C2) If $h : \boldsymbol{A} \to \boldsymbol{B}$ is a surjective homomorphism between $\boldsymbol{Q}$-algebras and $\Phi, \Psi \in \mathrm{Con}_{\mathbf{Q}}(\boldsymbol{A})$, then $\ker(h) +_{\boldsymbol{A}} [\Phi, \Psi]_{\boldsymbol{A}} = h^{-1}([\Theta^{\boldsymbol{B}}_{\boldsymbol{Q}}(h\Phi), \Theta^{\boldsymbol{B}}_{\boldsymbol{Q}}(h\Psi)]_{\boldsymbol{B}})$.

Theorem 2.2. *For any quasivariety $\boldsymbol{Q}$, if the equational commutator for $\mathbf{Q}$ satisfies (C1), then it satisfies (C2).* $\square$

Theorem 2.3. *Let $\mathbf{Q}$ be an RCM quasivariety. Then the equational commutator for $\mathbf{Q}$ and the commutator for $\mathbf{Q}$ in the sense of Kearnes-McKenzie* [5] *coincide.* $\square$

In the context of RCM quasivarieties we therefore speak of *the* commutator for $\mathbf{Q}$.

The crucial fact concerning the Kearnes-McKenzie commutator is that it satisfies (C1) for any RCM quasivariety $\mathbf{Q}$. In view of Theorems 2.1 and 2.3, we therefore get:

Corollary 2.4. *For every RCM quasivariety $\mathbf{Q}$ there exists a set $\Delta_0(x, y, z, w, \underline{u})$ of quaternary commutator equations for $\mathbf{Q}$ generating the commutator in the algebras of $\mathbf{Q}$.* $\square$

Generating sets of quaternary equations for the equational commutator are usually infinite and they involve parametric variables. More concrete forms of generating sets can be defined for quasivarieties with the relative shifting property. In some cases generating sets are finite:

Theorem 2.5. *Let $\mathbf{Q}$ be a quasivariety with the additive equational commutator. If $\mathbf{Q}$ is finitely generated, i.e., $\mathbf{Q} = \boldsymbol{SP}(\mathbf{K})$ for a finite set $\mathbf{K}$ of finite algebras, then the equational commutator of $\mathbf{Q}$ is generated by a finite set $\Delta_0(x, y, z, w, \underline{u})$ of quaternary commutator equations.* $\square$

Example. As the variety $\mathbf{BA}$ of Boolean algebras is congruence-distributive, the commutator of any two congruences on a Boolean algebra coincides with the meet of the two congruences.

Let $\alpha \approx \beta$ be the equation $(x \leftrightarrow y) \vee (z \leftrightarrow w) \approx \mathbf{1}$. ($\leftrightarrow, \vee$ and $\mathbf{1}$ stand for the Boolean operations of equivalence and join, respectively. $\mathbf{1}$ stands for the unit element.) $\alpha \approx \beta$ is a quaternary commutator equation for $\mathbf{BA}$.

The singleton set $\Delta(x, y, z, w, \underline{u}) := \alpha \approx \beta$ generates the (equational) commutator for $\mathbf{BA}$, i.e., for any $\boldsymbol{A} \in \mathbf{BA}$ and any $X, Y \subseteq A^2$,

$$[\Theta^{\boldsymbol{A}}_{\mathbf{Q}}(X), \Theta^{\boldsymbol{A}}_{\mathbf{Q}}(Y)]_{\boldsymbol{A}} = \Theta^{\boldsymbol{A}}_{\mathbf{Q}}(X) \cap \Theta^{\boldsymbol{A}}_{\mathbf{Q}}(Y) =$$
$$\Theta^{\boldsymbol{A}}_{\mathbf{Q}}(\langle \alpha(a, b, c, d), \beta(a, b, c, d)\rangle : \langle a, b\rangle \in X, \langle c, d\rangle \in Y). \qquad \square$$

3. Prime algebras

Let $\boldsymbol{A} \in \mathbf{Q}$, where $\mathbf{Q}$ is a quasivariety, and let Φ be a $\mathbf{Q}$-congruences on $\boldsymbol{A}$. Φ is *prime* (in the lattice $\mathrm{Con}_{\mathbf{Q}}(\boldsymbol{A})$) if, for any $\Phi_1, \Phi_2 \in \mathrm{Con}_{\mathbf{Q}}(\boldsymbol{A})$, $[\Phi_1, \Phi_2]_{\boldsymbol{A}} = \Phi$ implies $\Phi_1 = \Phi$ or $\Phi_2 = \Phi$. ($[\Phi_1, \Phi_2]_{\boldsymbol{A}}$ is the equational commutator of Φ_1, Φ_2 in $\boldsymbol{A}$ in the sense of $\mathbf{Q}$.)

$\boldsymbol{A} \in \mathbf{Q}$ is *prime* (in $\mathbf{Q}$) if $\mathbf{0_A}$ is prime in $\mathrm{Con}_{\mathbf{Q}}(\boldsymbol{A})$, i.e., $[\Phi_1, \Phi_2]_A = \mathbf{0_A}$ holds for no pair of nonzero congruences $\Phi_1, \Phi_2 \in \mathrm{Con}_{\mathbf{Q}}(\boldsymbol{A})$.

Theorem 3.1. *Let $\mathbf{Q}$ be quasivariety with the additive equational commutator and a generating set $\Delta(x, y, z, w, \underline{u})$. Suppose $\boldsymbol{A} \in \mathbf{Q}$. The following conditions are equivalent:*

(i) *$\boldsymbol{A}$ is prime.*

(ii) $\boldsymbol{A} \models (\forall xyzw)((\forall \underline{u}) \wedge \Delta(x, y, z, w, \underline{u}) \leftrightarrow x \approx y \vee z \approx w)$. $\square$

The basic result:

Theorem 3.2. *Let $\mathbf{Q}$ be a quasivariety with the additive equational commutator. (In particular, let $\mathbf{Q}$ be RCM.) Then*

(1) *The class* $\boldsymbol{SP}(\boldsymbol{Q}_{PRIME})$ *is the largest RCD quasivariety included in* $\mathbf{Q}$.

(2) $\mathbf{Q}_{PRIME}$ *coincides with the class of all relatively finitely subdirectly irreducible algebras in* $\boldsymbol{SP}(\boldsymbol{Q}_{PRIME})$.

(3) $\boldsymbol{SP}(\boldsymbol{Q}_{PRIME})$ *is axiomatized by any basis for* $\mathbf{Q}$ *augmented with a single quasi-identity.* □

Remarks.1. Let $\mathbf{R}$ be the variety of rings. (The existence of unit is not assumed.) $\mathbf{R}$ is congruence permutable and hence congruence modular. Dziobiak [2] describes RCD quasivarieties contained in $\mathbf{R}$.

2. Kearnes [4] contains various characterizations of RCD subquasivarieties of congruence modular *varieties*. □

References

[1] J. Czelakowski *General theory of the commutator for deductive systems. Part I. Basic facts*, Studia Logica **83**, 183-214. [2006]

[2] W. Dziobiak *Relative congruence distributivity within quasivarieties of nearly associative* Φ*-algebras*, Fundamenta Mathematicae **135**, No. 2, 77-95. [1990]

[3] R. Freese and R. McKenzie "Commutator Theory for Congruence Modular Varieties" , London Mathematical Society Lecture Note Series 125, Cambridge University Press, Cambridge-New York. [1987]

[4] K. Kearnes *Relatively congruence distributive subquasivarieties of a congruenece modular variety*, Bulletin of the Australian Mathematical Society **41**, 87-96. [1990]

[5] K. Kearnes and R. McKenzie *Commutator theory for relatively modular quasivarieties*, Transactions of the American Mathematical Society **331**, No. 2, 465-502. [1992]