Semigroups and Semirings of Boolean Operations

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Let $O^n(A)$ be the set of all n-ary operations defined on the set A with $|A| \geq 2$ and let $O(A) := \bigcup\limits_{n\geq 1} O^n(A)$ be the set of all operations defined on A. If $f \in O^n(A)$ and $g_1, \ldots, g_n \in O^m(A)$, we define the superposition operation S^n_m by

$$S^n_m(f, g_1, \ldots, g_n)(a_1, \ldots, a_m) := f(g_1(a_1, \ldots, a_m), \ldots, g_n(a_1, \ldots, a_m))$$

for all $a_1, \ldots, a_m \in A$. If $n = m$ we will write for short S^n instead of S^n_n. A subset $C \subseteq O(A)$ is called a clone if C is closed under all operations S^n_m and contains all projections $e^n_i : A^n \to A, i \leq n$, defined by $e^n_i(a_1, \ldots, a_n) := a_i$ for all $a_1, \ldots, a_n \in A$. Clones can be regarded as multi-based algebras $\mathcal{C} := ((C^{(n)})_{n\geq 1}, (S^n_m)_{m,n\geq 1}, (e^n_i)_{n\geq 1, i\leq n})$ where $C^{(n)}$ is the set of all n-ary operations belonging to C.

If $f, g \in O^n(A)$ we define a binary operation $+$ by

$$f + g = S^n(f, g, \ldots, g)$$

From the super-associativity of S^n there follows that this operation is associative and we obtain a semigroup $\mathcal{O}^n(A) = (O^n(A); +)$.

Since any clone can be regarded as the universe of a subalgebra of

$$((O^n(A)_{n\geq 1}, (S^n_m)_{m,n\geq 1}, (e^n_i)_{n\geq 1, i\leq n}),$$

all clones of operations defined on the set A form a complete algebraic lattice. If A is finite, this lattice is atomic and dually atomic with finitely many atoms and dual atoms (see [1]). In the case that A is the two-element set $\{0, 1\}$, this lattice is called lattice of Boolean clones and is denoted by $\mathcal{L}_2$. It was first described by E. L. Post in 1941(1921) ([2], [3]), and is sometimes also called Post's lattice. The lattice of all Boolean clones is countably infinite and every clone in the lattice is finitely generated.

The notation which is used in the diagram of Figure 1 goes partly back to E. L. Post.

For $|A| > 2$ the lattice of all subclones of $O(A)$ is uncountably infinite and not fully described. There are subclones of $O(A), |A| > 2$ which are not finitely generated. Because of these difficulties the idea to try another classification of the elements of $O(A)$ is very seducing. We propose instead of clones to consider subsemigroups of the semigroup $(O^n(A); +), n \geq 1$. It is very natural to start this project with $A = \{0, 1\}$ and then to continue with more complicated situations when $|A| > 2$.

Let $\mathcal{P}(O(A))$ be the power set of the set of all operations defined on the set A. We introduce the following binary operations on $\mathcal{P}(O(A))$: an addition $\oplus$ which is defined as set-theoretical union and a multiplication $\otimes$ defined by

$$X \otimes Y := \{f + g | f \in X, g \in Y\}, X, Y \in \mathcal{P}(O(A)).$$

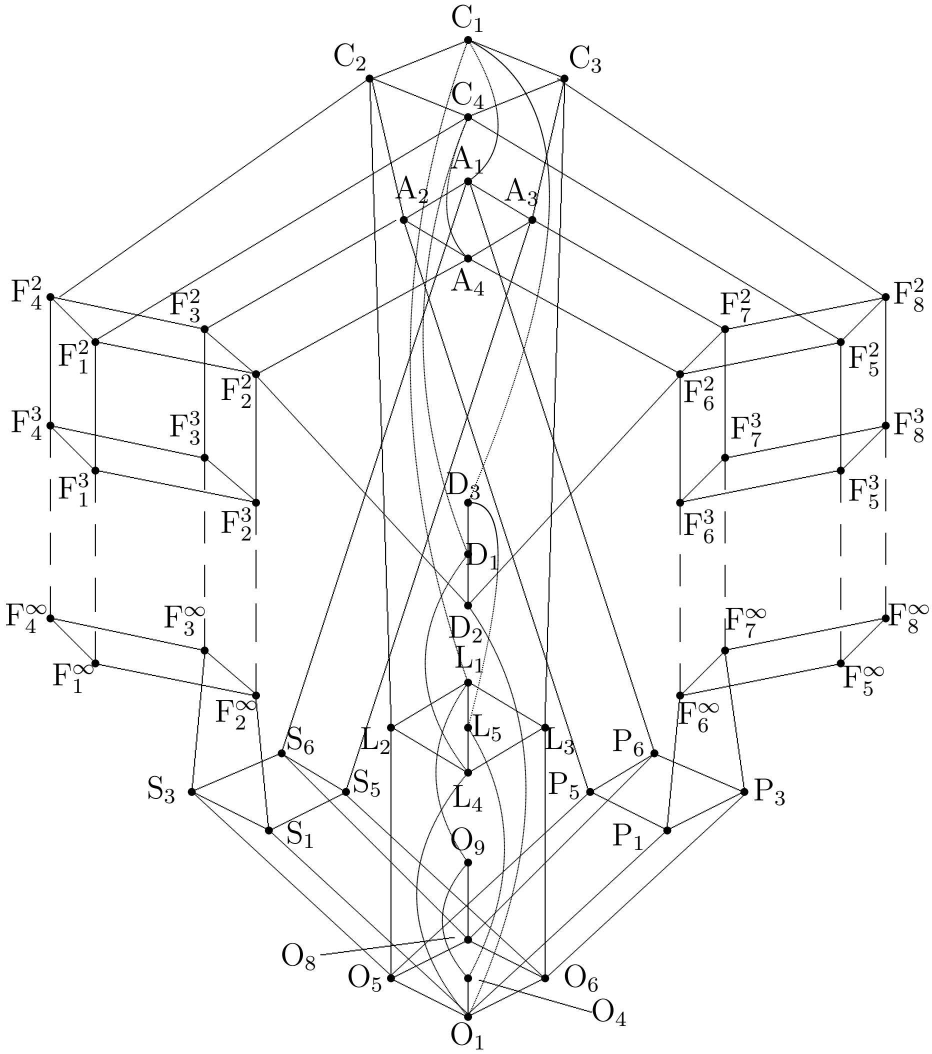

FIGURE 1. Post's Lattice

Then we obtain a semiring $(\mathcal{P}(O(A)); \oplus, \otimes)$ and we can investigate semiring-theoretical properties of this semiring and its subsemirings.

For $A = \{0, 1\}$ some typical results of this research are:

(1) Determination of the order of each $f \in O^n(A)$.
(2) Determination of all idempotent and of all regular elements in $(O^n(A); +)$.
(3) Determination of subsemigroups of $(O^n(A); +)$ with particular properties (right-zero semigroups, left-zero semigroups, rectangular bands, semilattices).
(4) Description of all kinds of semigroups which occur as subsemigroups of $(O^n(A); +)$.
(5) Characterization of all regular, k-regular and completely regular subsemirings of $(\mathcal{P}(O(A)); \oplus, \otimes)$.

References

[1] Lau, D. : *Function Algebras on Finite Sets,* Springer, 2006.

[2] Post, E. L. : *Introduction to a General Theory of Elementary propositions,* Amer. J. Math. 43(1921).

[3] Post, E. L. : *The two-valued iterative systems of mathematical logic,* Ann. Math. Studies 5, Princeton Univ. Press, 1941.