

The algebraic structure of quasigroup formulas

Jan Gałaszka

e-mail: jan.galuszka@polsl.pl

Silesian University of Technology, Gliwice, Poland

1. INTRODUCTION

In this presentation, an algebraic structure on the set of quasigroup formulas is introduced. In our considerations an important role is played by ‘power terms’ and ‘power identities’ (the definitions are recalled in Section 3). Some earlier results indicating connections of some kinds of ‘power identities’ with quasigroup properties were presented in [4] and [2]. In the main theorem of this paper we prove that the set of quasigroup formulas is naturally equipped with a lattice structure isomorphic to the lattice of closed Steinitz numbers with the divisibility relation.

Recall that the idea of enhanced natural numbers now known as Steinitz numbers was introduced by E. Steinitz [10]. This concept was also considered by A. Robinson [9]. The notion of closed Steinitz numbers was proposed in [6].

The presentation is closely connected with the results presented in [5], [6] and [7]. Algebraic notions and notations used in the presentation are standard and similar to those used in [3] or [8]. Some basic notions specific to quasigroup theory are analogous to those introduced in [1].

2. NOTATIONS AND TERMINOLOGY

This section provides some technical preliminaries. The notation and terminology is briefly reviewed.

Recall that: By a groupoid is meant an algebra $\mathfrak{G} = (G, \cdot)$ of type (2) with universe (base set, underlying set) G and binary operation

$$\cdot : G \times G \longrightarrow G \quad ((x, y) \mapsto xy).$$

Recall that in groupoid theory, by a *right* (resp. *left*) *quasigroup* we mean a groupoid \mathfrak{G} such that for all $a, b \in G$ the equation $xa = b$ (resp. $ax = b$) has a unique solution. By \mathcal{QG}^* will be denoted the class of right quasigroups. By a *one-sided quasigroup* we mean a groupoid which is a right or left quasigroup. By a *quasigroup* we mean a groupoid which is a right and left quasigroup simultaneously. If a groupoid \mathfrak{G} is a right quasigroup then its dual groupoid is a left quasigroup and vice versa. This duality establishes a symmetrical correspondence between ‘right’ and ‘left’ versions of statements (to every theorem in the right version corresponds its dual left version and vice versa). Therefore, for conciseness we formulate almost all statements below in one (right) version only.

Using combinatorics terminology we can say that a groupoid \mathfrak{G} is a right quasigroup if and only if for every $u \in G$ the right translation

$$(1) \quad r_u^{\mathfrak{G}} : G \longrightarrow G \quad (x \mapsto xu)$$

is a bijection. This interpretation will be useful for us.

3. MAIN RESULTS

Recall that *power terms* (more precisely *right-power terms*) are terms defined inductively as follows:

$$xy^0 := x, \quad xy^{n+1} := (xy^n)y,$$

where $n \in \mathbb{N}$ (*left-power terms* ${}^n yx$ can be defined similarly). With the family of power terms is associated a family of identities. For each $n \in \mathbb{N}_1$ we have the *power identity*

$$(*^n) \quad xy^n = x.$$

Power terms and power identities play an important role in [5], [6] and [7].

For some quasigroup aspects of power identities in commutative and idempotent groupoids see [4]. For some quasigroup aspects of power identities in finite groupoids see [2].

It is clear from the logical point of view that the formulas $*^n$ are abbreviated forms of the following formulas:

$$(*^{n'}) \quad \varphi_n := \ulcorner \forall x, y \ xy^n = x \urcorner.$$

In [5] as a generalization of $*^n$ we propose the following *torsion formulas* (more precisely *right-torsion formulas*):

$$(*^t) \quad \varphi_t := \ulcorner \forall x, y \ \exists n \in \mathbb{N}_1 \ xy^n = x \urcorner.$$

In the following considerations we use Steinitz numbers. A slightly reformulated construction of these numbers is proposed in [[6], Section 3] (for the original construction see [[10], p. 250]). We introduce a family of new formulas as follows: Let us denote by \mathbb{S} the set of Steinitz numbers. Let $s \in \mathbb{S}$. The following formulas can be seen as generalizations of the identities $*^n$ for Steinitz numbers:

$$(*^s) \quad \varphi_s := \ulcorner \exists n \in \mathbb{N}_1 \ \forall x, y \ (n|s \wedge xy^n = x) \urcorner.$$

Associated with the torsion formula for $s \in \mathbb{S}$, we propose the following formulas:

$$(*^t_s) \quad \varphi_{s,t} := \ulcorner \forall x, y \ \exists n \in \mathbb{N}_1 \ (n|s \wedge xy^n = x) \urcorner.$$

Let us introduce the following notations: $F_{\mathbb{N}_1} := \{\varphi_n \mid n \in \mathbb{N}_1\}$, $F_{\mathbb{S}} := \{\varphi_s \mid s \in \mathbb{S}\}$, $F_{\mathbb{S},t} := \{\varphi_{s,t} \mid s \in \mathbb{S}\}$, $F_t := \{\varphi_t\}$. Lemma 1 below collects some results presented in [[5], Theorem 9], [[5], Theorem 15] and [[7], Section 2].

Lemma 1. *If a formula φ where $\varphi \in F_{\mathbb{N}_1} \cup F_{\mathbb{S}} \cup F_{\mathbb{S},t} \cup F_t$ is satisfied in a groupoid \mathfrak{G} then \mathfrak{G} is a right quasigroup.*

Evidently there are groupoids which are right quasigroups without satisfying any of $*^n$, $*^t$, $*^s$ and $*^t_s$. The natural question is: what formulas satisfied in groupoids force a one-sided quasigroup structure on them.

Let $F_{\mathcal{Q}\mathfrak{G}^*} := F_{\mathbb{N}_1} \cup F_{\mathbb{S}} \cup F_{\mathbb{S},t} \cup F_t \cup F_{\infty}$ where $F_{\infty} := \{\varphi_{\infty}\}$ and

$$(*^{\infty}) \quad \varphi_{\infty} := \ulcorner \forall x, y \ \left(\begin{array}{l} (\exists n \in \mathbb{N}_1 \ xy^n = x) \\ \vee \\ ((\forall n \in \mathbb{N}_1 \ xy^n \neq x) \wedge (\exists z \ zy = x)) \end{array} \right) \urcorner.$$

Proposition 2.

$$(2) \quad \mathfrak{G} \in \mathcal{QG}^* \iff \exists \varphi \in F_{\mathcal{QG}^*} \ \mathfrak{G} \models \varphi.$$

Formulas that are members of $F_{\mathcal{QG}^*}$ are named *one-sided (right) quasigroup formulas*. Let φ be a formula (in a language adequate for groupoid theory). Define

$$\mathcal{Q}_\varphi := \{\mathfrak{G} \mid \mathfrak{G} \models \varphi\}.$$

As a consequence of Proposition 2 we have

Corollary 3.

$$(3) \quad \mathcal{QG}^* = \bigcup \{\mathcal{Q}_\varphi \mid \varphi \in F_{\mathcal{QG}^*}\}$$

On the set $F_{\mathcal{QG}^*}$ we will consider a binary relation \leq defined as follows: for $\varphi, \psi \in F_{\mathcal{QG}^*}$,

$$(4) \quad \varphi \leq \psi \iff \mathcal{Q}_\varphi \subseteq \mathcal{Q}_\psi.$$

With every right quasigroup \mathfrak{G} there is associated a Steinitz number denoted by $\langle \mathfrak{G} \rangle$ and called the *code number* of \mathfrak{G} (see [[6], Section 5.1.3]). More precisely, we have the function

$$(5) \quad h: \mathcal{QG}^* \longrightarrow \bar{\mathbb{S}} \ (x \mapsto \langle x \rangle).$$

Recall that the closed Steinitz numbers with the divisibility relation form a complete lattice $\bar{\mathfrak{S}} = (\bar{\mathbb{S}}, |)$ (cf. [[6], Section 3]). Let $\varphi \in F_{\mathcal{QG}^*}$. Define

$$(6) \quad \langle \varphi \rangle := \sup \{ \langle x \rangle \mid x \in \mathcal{Q}_\varphi \}.$$

The Steinitz number $\langle \varphi \rangle$ will be called the *code number* of the (right) quasigroup formula φ .

The following theorem is true:

Theorem 4.

- (i) *The relational system $\mathfrak{F}_{\mathbb{N}_1} = (F_{\mathbb{N}_1}; \leq)$ forms a lattice isomorphic to the lattice $\mathfrak{N}_1 = (\mathbb{N}_1; |)$ of positive integers with the divisibility relation.*
- (ii) *The relational systems $\mathfrak{F}_{\mathbb{S}} = (F_{\mathbb{S}}; \leq)$ and $\mathfrak{F}_{\mathbb{S},t} = (F_{\mathbb{S},t}; \leq)$ form complete lattices both isomorphic to the lattice $\mathfrak{S} = (\mathbb{S}; |)$ of Steinitz numbers with the divisibility relation.*
- (iii) *The relational system $\bar{\mathfrak{F}}_{\mathbb{S},t} = (\bar{F}_{\mathbb{S},t}; \leq)$ where $\bar{F}_{\mathbb{S},t} := F_{\mathbb{S},t} \cup \{\varphi_\infty\}$ forms a lattice isomorphic to the lattice $\bar{\mathfrak{S}} = (\bar{\mathbb{S}}; |)$ of closed Steinitz numbers with the divisibility relation.*

REFERENCES

- [1] V.D. Belousov, *Foundations of the Theory of Quasigroups and Loops* (Russian), Izdat. "Nauka", Moscow, 1967.
- [2] D. E. Bryant and S. Oates-Williams, Constructing identities for finite quasigroups, *Comm. Algebra* 22 (1994), no. 5, 1783–1795.
- [3] S. Burris and H. P. Sankappanavar, *A course in universal algebra*, Springer, New York, 1981.
- [4] J. Dudek, On binary polynomials in idempotent commutative groupoids, *Fund. Math.* 120 (1984) 187–191.

- [5] J. Gałaszka, Groupoids with quasigroup and Latin square properties, *Discrete Math.* 308 (2008), no. 24, 6414–6425.
- [6] J. Gałaszka, Codes of groupoids with one-sided quasigroup conditions, *Algebra Discrete Math.* (2009), no. 2, 27–44.
- [7] J. Gałaszka, Lattices of classes of groupoids with one-sided quasigroup conditions, *Algebra Discrete Math.* (2010), no. 1.
- [8] G. Grätzer, *Universal Algebra*, Van Nostrand, 1979.
- [9] A. Robinson, Nonstandard arithmetic, *Bull. Amer. Math. Soc.* 73 (1967), 818–843.
- [10] E. Steinitz, Algebraische Theorie der Körper, *J. reine angew. Math.* 137 (1910), 167–309.