# The possible values of critical points between varieties of lattices

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An *algebra* (in the sense of universal algebra) is a nonempty set $A$ endowed with a collection of maps ("operations") from finite powers of $A$ to $A$. One of the most fundamental invariants associated with an algebra $A$ is the lattice $\operatorname{Con} A$ o f all *congruences* of $A$, that is, the equivalence relations on $A$ compatible with all the operations of $A$. It is often more convenient to work with the $(\vee, 0)$-semilattice $\operatorname{Con_c} A$ of all *finitely generated congruences* of $A$. We set

$$\operatorname{Con_c} \mathcal{V} = \{ S \mid (\exists A \in \mathcal{V})(S \cong \operatorname{Con_c} A) \}, \quad \text{for every class } \mathcal{V} \text{ of algebras,}$$

and we call $\operatorname{Con_c} \mathcal{V}$ the *compact congruence class of* $\mathcal{V}$. This object is especially studied for $\mathcal{V}$ a *variety* (or *equational class*) *of algebras*, that is, the class of all algebras that satisfy a given set of *identiti es* (in a given similarity type).

For a variety $\mathcal{V}$, two fundamental questions arise:

(**Q1**) Is $\operatorname{Con_c} \mathcal{V}$ determined by a reasonably small fragment of itself—for example, is it determined by its *finite* members?
(**Q2**) In what extent does $\operatorname{Con_c} \mathcal{V}$ characterize $\mathcal{V}$?

While both questions remain largely mysterious in full generality, partial answers are known in a number of situations. Here is an example that illustrates the difficulties underlying question (**Q1**) above. We are working with *varieties of lattices* . Denote by $\mathcal{D}$ the variety of all distributive lattices and by $\mathcal{M}_3$ the variety of lattices generated by the five-element modular nondistributive lattice $M_3$. The finite members of both $\operatorname{Con_c} \mathcal{D}$ and $\operatorname{Con_c} \mathcal{M}_3$ are exactly the finite Boolean latti ces. It is harder to prove that there exists a *countable* lattice $K \in \mathcal{M}_3$ such that $\operatorname{Con_c} K \ncong \operatorname{Con_c} D$ for any $D \in \mathcal{D}$. This can be obtained easily from the results of [1], but a direct construction is also possible, as follows: denote by $S$ a copy of the two-atom Boolean lattice in $M_3$; let $K$ be the lattice of all eventually constant sequences of elements of $M_3$ such that the limit belongs to $S$. (This construction is a precursor of the *condensates* introduced in [1].)

This suggests a way to *measure* the "containment defect" of $\operatorname{Con_c} \mathcal{V}$ into $\operatorname{Con_c} \mathcal{W}$, for varieties (not necessarily in the same similarity type) $\mathcal{V}$ and $\mathcal{W}$. The *critical point* between $\mathcal{V}$ and $\mathcal{W}$ is defined as $\operatorname{crit}(\mathcal{V}; \mathcal{W}) = \min \{ \operatorname{card} S \mid S \in (\operatorname{Con_c} \mathcal{V}) - (\operatorname{Con_c} \mathcal{W}) \}$ if $\operatorname{Con_c} \mathcal{V} \nsubseteq \operatorname{Con_c} \mathcal{W}$, otherwise we put $\operatorname{crit}(\mathcal{V}; \mathcal{W}) = \infty$. The example above shows that

$$\operatorname{crit}(\mathcal{M}_3; \mathcal{D}) = \aleph_0.$$

Using varieties generated by finite non-modular lattices it is easy to construct *finite* critical points. By using techniques from *infinite combinatorics*, introduced in [8], Ploščica found in [4], [5] varieties of bounded lattices with critical point $\aleph_2$; the bounds of those examples were subsequently removed, in [2]. For example, if we denote by $\mathcal{M}_n$ the variety generated by the lattice of length 2 with $n$ atoms, then

$$\operatorname{crit}(\mathcal{M}_m; \mathcal{M}_n) = \aleph_2 \quad \text{if } m > n \geq 3.$$

In [1] we find a pair of finitely generated modular lattice varieties with critical point $\aleph_1$, thus answering a 2002 question of Tůma and Wehrung from [7].

In "crossover" contexts (varieties on different similarity types), the situation between lattices and congruence-permutable varieties (say, *groups* or *modules*) is quite instructive. The five-element lattice $M_3$ is isomorphic to the congrue nce lattice of a group (take the Klein group) but not to the congruence lattice of any lattice (for it fails distributivity), and it is easily seen to be the smallest such example. Thus, if we denote by $\mathcal{L}$ the variety of all lattices and by $\mathcal{G}$ the va riety of all groups, then

$$\operatorname{crit}(\mathcal{G}; \mathcal{L}) = 5.$$

Much harder techniques, also originating from [8], are applied in [6], yielding the following result:

$$\operatorname{crit}(\mathcal{L}; \mathcal{G}) = \aleph_2.$$

A general "crossover result", proved in [1], is the following *Dichotomy Theorem*:

**Theorem 1.** *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of algebras, with $\mathcal{V}$ locally finite and $\mathcal{W}$ finitely generated congruence-distributive* (for example, any variety of lattices)*. If $\operatorname{Con_c} \mathcal{V} \nsubseteq \operatorname{Con_c} \mathcal{W}$, then $\operatorname{crit}(\mathcal{V}; \mathcal{W}) < \aleph_\omega$.*

In [3], we use techniques of *category theory* to extend this Dichotomy Theorem to much wider contexts, in particular to *congruence-modular* (instead of congruence-distributive) varieties and even to relative congruence classes of *quasivarieties*. Examples of congruence-modular varieties are varieties of *groups* (or even *loops*) or *modules*.

Now comes another mystery. We do not know any critical point, between varieties of algebras with (at most) countable similarity types, equal to $\aleph_3$, or $\aleph_4$, and so on: all known critical points are either below $\aleph_2$ or equal to $\infty$ . How general is that phenomenon?

One half of our main result, is the following:

**Theorem 2.** *Let $\mathcal{V}$ and $\mathcal{W}$ be lattice varieties such that every simple member of $\mathcal{W}$ contains a prime interval. If $\operatorname{Con_c} \mathcal{V} \nsubseteq \operatorname{Con_c} \mathcal{W}$, then $\operatorname{crit}(\mathcal{V}; \mathcal{W}) \leq \aleph_2$.*

In particular, this result gives a solution to Question (**Q1**) for varieties of lattices where every simple member contains a prime interval. It turns out that the other half of our main result also solves Question (**Q2**) for those varieties, by proving that $\operatorname{Con_c} \mathcal{V} \subseteq \operatorname{Con_c} \mathcal{W}$ occurs only in the trivial cases, namely $\mathcal{V}$ is contained in either $\mathcal{W}$ or its dual.

Our main idea is the following. It is well-known that the assignment $A \mapsto \operatorname{Con_c} A$ can be extended, in a standard way, to a *functor*, from all algebras of a given similarity type with their homomorphisms, to the category of all $(\vee, 0)$-semilattices and $(\vee, 0)$-homomorphisms: for a homomorphism $f \colon A \to B$ of algebras, $\operatorname{Con_c} f$ sends any compact congruence $\alpha$ of $A$ to the congruence generated by all pairs $(f(x), f(y))$ where $(x, y) \in \alpha$. Given a finite bounded lattice $L$, we construct a diagram of $(\vee, 0)$-semilattices lif table in a variety of lattices (or bounded lattices) $\mathcal{V}$ if and only if either $L$ or its dual belongs to $\mathcal{V}$. Then, using a *condensate* [[3], Section 3-1], we construct a $(\vee, 0)$-semilattice liftable in $\mathcal{V}$ if and only if either $L \in \mathcal{V}$ or $L^{\mathrm{d}} \in \mathcal{V}$.

The main idea is to start with the *chain diagram* $\vec{A}$ *of* $L$. A precursor of that diagram can be found in [[2], Section 3-1]. The chain diagram can be described in the following way. The arrows are the inclusion maps. At the bottom of t he diagram we put $\{0, 1\}$, the sublattice of $L$ consisting of the bounds of $L$. On the next level we put all three-element and four-element chains of $L$ with extremities 0 and 1, over two chains we put the sublattice of $L$ generated by those tw o chains, finally at the top we put the lattice $L$ itself.

The diagram $\mathrm{Con_c} \circ \vec{A}$ is liftable in any variety that contains either $L$ or its dual; we do not know any counterexample of the converse yet. However, given a lifting $\vec{B}$ of $\mathrm{Con_c} \circ \vec{A}$, with all morphisms of $\vec{B}$ being inclusion m aps, if all lattices of $\vec{B}$ that correspond to chains in $\vec{A}$ contain "congruence chains" with the same extremities, we also require those chains be "direct". For example, if $A_i = \{0, x, 1\}$ is a chain, then $B_i = \{u, y, v\}$ is also a chain (with $u < y < v$), the congruence $\Theta_{A_i}(0, x)$ corresponds to the congruence $\Theta_{B_i}(u, y)$, and the congruence $\Theta_{A_i}(x, 1)$ corresponds to the congruence $\Theta_{B_i}(y, v)$. Under these assumptions we construct a sublattice of the top me mber of $\vec{B}$ isomorphic to $L$.

The second step of our construction is to expand the chain diagram in order to force congruence chains to be direct. By gluing the chain diagram of $L$ and copies of a "directing" diagram, we obtain a diagram $\vec{A}'$ such that whenever $\vec{B}$ is a li fting of $\mathrm{Con_c} \circ \vec{A}'$ with enough congruence chains, then $L$ embeds into a quotient of some lattice of $\vec{B}$ or its dual.

The third step is to ensure the existence of enough congruence chains. We can construct a diagram $\vec{A}''$ such that if $\mathrm{Con_c} \circ \vec{A}''$ is liftable in some variety $\mathcal{W}$, then either $L \in \mathcal{W}$ or $L \in \mathcal{W}^{\mathrm{d}}$. For this step, we need a variety $cW$ where every simple lattice has a prime interval (i.e., elements $u$ and $v$ such that $u \prec v$).

The last step is to use a *condensate* construction [[3], Section 3-1] on $\vec{A}''$ to obtain a lattice $B$ of cardinality $\aleph_2$ such that $\mathrm{Con_c}\, B$ is liftable in $\mathcal{W}$ if and only if $\mathrm{Con_c} \circ \vec{A}''$ has a "partial lifting" in $\mathcal{W}$. Hence $\mathrm{Con_c}\, B$ is liftable in $\mathcal{W}$ if and only if either $L \in \mathcal{W}$ or $L \in \mathcal{W}^{\mathrm{d}}$.

The chain diagram also makes it possible to prove that if there is a functor $\Psi \colon \mathcal{V} \to \mathcal{W}$ such that $\mathrm{Con_c} \circ \Psi$ is equivalent to $\mathrm{Con_c}$, then either $\mathcal{V} \subseteq \mathcal{W}$ or $\mathcal{V} \subseteq \mathcal{W}^{\mathrm{d}}$. The functor $\Psi$ itself does not need to be equivalent to either inclusion or dualization, however we prove that this holds "up to congruence-preserving extensions".

## References


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