Dependence spaces

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1. Introduction

This is a continuation of my lecture presented on 77th Workshop on General Algebra, 24th Conference for Young Algebraists in Potsdam (Germany) on 21st March 2009. The Steinitz exchange lemma is a basic theorem in linear algebra used, for example, to show that any two bases for a finite-dimensional vector space have the same number of elements. The result is named after the German mathematician Ernst Steinitz.

We present another proof of the result of N.J.S. Hughes [4] on Steinitz' exchange theorem for infinite bases in connection with the notions of transitive dependence, independence and dimension as introduced in [2] and [8]. In our proof we assume Kuratowski-Zorn's Lemma, as a requirement pointed in [3]. According to F. Gécseg, H. Jürgensen [3] the result which is usually referred to as the "Exchange Lemma", states that for transitive dependence, every independent set can be extended to form a basis. Our aim is to discuss some interplay between the discussed notion of [4] and [3].

We use a modification of the the notation of [4], [5] and [3]:
$a, b, c, ..., x, y, z, ...$ (with or without suffices) to denote the elements of $\mathbf{S}$ and $A, B, C, ..., X, Y, Z, ...$ for subsets of $\mathbf{S}$, $\mathbb{X}$, $\mathbb{Y}$,... denote a family of subsets of $\mathbf{S}$, n is always a positive integer.

$A \cup B$ denotes the union of sets A and B, $A + B$ denotes the disjoint union of A and B, $A - B$ denotes the difference of A and B, i.e. is the set of those elements of A which are not in B.

2. Dependent and independent sets

The following definition is due to N.J.S. Hughes, invented in 1962 in [4]:

Definition 1. A set $\mathbf{S}$ is called a *dependence space* if there is defined a set Δ, whose members are finite subsets of $\mathbf{S}$, each containing at least 2 elements, and if the Transitivity Axiom is satisfied.

Definition 2. A set A is called *directly dependent* if $A \in \Delta$.

Definition 3. An element x is called *dependent on* A and is denoted by $x \sim \Sigma A$ if either $x \in A$ or if there exist distinct elements $x_0, x_1, ..., x_n$ such that

$$(1)\ \{x_0, x_1, ..., x_n\} \in \Delta$$

where $x_0 = x$ and $x_1, ..., x_n \in A$

and *directly dependent* on $\{x\}$ or $\{x_0, x_1, ..., x_n\}$, respectively.

Definition 4. A set A is called *dependent* if (1) is satisfied for some distinct elements $x_0, x_1, ..., x_n \in A$, and otherwise A is *independent*.

Definition 5. If a set A is *independent* and for any $x \in \mathbf{S}$, $x \sim \Sigma A$, i.e. x is dependent on A, then A is called a *basis of* $\mathbf{S}$.

Definition 6. TRANSITIVITY AXIOM:

If $x \sim \Sigma A$ and for all $a \in A$, $a \sim \Sigma B$, then $x \sim \Sigma B$.

Similar definition of a *dependence* D was introduced in [2] and [3]. Following ideas of [2] and [3] we accept the definition of the *span* $< X >$ of a subset X of $\mathbf{S}$ to be the set of all elements of $\mathbf{S}$ which depends on X. A subset X of $\mathbf{S}$ is called *closed* if $X =< X >$, and a dependence D is called *transitive*, if $< X >=<< X >>$, for all subsets X of $\mathbf{S}$.

First we note, that the notion of *transitive dependence* of [2] and [3] is equivalent to those of [4].

Theorem 7. *Given a dependence space* $\mathbf{S}$ *satisfying the Transitivity Axiom in the sense of* [4]. *Then* $< X >=<< X >>$, *for all subsets* X *of* $\mathbf{S}$, *i.e. dependence is transitive in the sense of* [2] *and* [3]. *And vice versa, if a dependence is transitive in the sense of* [2] *and* [3], *then the Transitivity Axiom of* [4] *is satisfied.*

Proof

Let a transitive axiom of [4] be satisfied in $\mathbf{S}$ and let $X \subseteq \mathbf{S}$.
$X \subseteq < X >$ and therefore $< X > \subseteq << X >>$, by Remark 3.6 of [2]. Now let $x \in << X >>$, i.e. $x \sim \Sigma < X >$. But $y \sim \Sigma X$, for all $y \in < X >$. Thus by the Transitivity Axiom $x \sim \Sigma X$, i.e. $x \in < X >$. We get that $< X >=<< X >>$.

Now, assume that a dependence is transitive in the sense of [2], [3], i.e. $\mathbf{S}$ be a transitive dependence space, i.e. $<< X >>=< X >$ for all $X \subseteq \mathbf{S}$. Let $x \sim \Sigma A$ and for all $a \in A$, $a \sim \Sigma B$. Thus $x \in A$ or the set $\{x\} \cup A$ is dependent, i.e. $x \in < A >$. Moreover, $a \in < B >$, for all $a \in A$. Therefore $\{a\} \subseteq < B >$, for all $a \in A$, i.e. $< a > \subseteq << B >>=< B >$, by the Remark 3.6 of [3]. In consequence: $A = \bigcup\{a : a \in A\} \subseteq \bigcup\{< a >: a \in A\} \subseteq < B >$ and thus:
$< A > \subseteq << B >>=< B >$. We get that $x \in < B >$, i.e. $x \sim \Sigma B$, i.e. the Transitivity Axiom is satisfied. □

3. Po-set of independent sets

In this section we note, that the following well known properties (see [2] or [3]) are satisfied in a dependence space:

Proposition 8. *(2) Any subset of an independent set A is independent.*

(3) A basis is a maximal independent set of $\mathbf{S}$ *and vice versa.*

(4) A basis is a minimal subset of $\mathbf{S}$ *which spans* $\mathbf{S}$ *and vice versa.*

(5) The family $(\mathbb{X}, \subseteq)$ *of all independent subsets of* $\mathbf{S}$ *is partially ordered by the set-theoretical inclusion. Shortly we say that* $\mathbb{X}$ *is an ordered set (a po-set).*

(6) Any superset of a dependent set of $\mathbf{S}$ *is dependent.*

Many examples of the notions above may be found in [1]–[4] and [8].

Following K. Kuratowski and A. Mostowski [7] p. 241, a po-set $(\mathbb{X}, \subseteq)$ is called *closed* if for every chain of sets $\mathbb{A} \subseteq P(\mathbb{X})$ there exists $\cup\mathbb{A}$ in $\mathbb{X}$, i.e. $\mathbb{A}$ has the supremum in $(\mathbb{X}, \subseteq)$.

Theorem 9. *The po-set* $(\mathbb{X}, \subseteq)$ *of all independent subsets of* $\mathbf{S}$ *is closed.*

Theorem 10. *The po-set* $(\mathbb{X}, \subseteq)$ *of all independent subsets of* $\mathbf{S}$ *is an algebraic closure system.*

Proof

Let $\mathbb{A}$ be a directed family of independent subsets of $\mathbf{S}$, i.e. $\mathbb{A} \subseteq P(\mathbb{X})$, and for all $A, B \in \mathbb{A}$ there exists a $C \in \mathbb{A}$ such that $(A \subseteq C)$ and $(B \subseteq C)$. We show that the set $\cup\mathbb{A}$ is independent. Otherwise there exist elements $\{x_0, x_1, ..., x_n\} \in \Delta$ such that $x_i \in \cup\mathbb{A}$, for $i = 0, ..., n$. Therefore there exists a set $C \in \mathbb{A}$ such that $x_i \in C$ for all $i = 0, ..., n$.
We conclude that C is dependent, $C \in \mathbb{A}$, a contradiction. □

4. Steinitz' exchange theorem

A transfinite version of the Steinitz Exchange Theorem, provides that any independent subset injects into any generating subset. For more information on the role of Steinitz papers consult the book chapter *400 Jahre Moderne Algebra*, of [1]. The following is a generalization of Steinitz' Theorem originally proved in 1913 and then in [4]–[5]:

Theorem 11. *If* A *is a basis and* B *is an independent subset (of a dependence space* $\mathbf{S}$*). Then assuming Kuratowski-Zorn Maximum Principle, and the Transitivity Axiom, there is a definite subset* A' *of* A *such that the set* $B + (A - A')$ *is also a basis of* $\mathbf{S}$.

Proof If B is a basis then B is a maximal independent subset of $\mathbf{S}$ and $A' = A$ is clear.

Assume that A is a basis and B is an independent subset (of the dependence space $\mathbf{S}$). Consider $\mathbb{X}$ to be the family of all independent subsets of $\mathbf{S}$ containing B and contained in $A \cup B$. then $(\mathbb{X}, \subseteq)$ is well ordered and closed. Therefore assuming Kuratowski-Zorn Maximal Principle [6] there exists a maximal element of $\mathbb{X}$.
We show that this maximal element $X \in \mathbb{X}$ is a basis of $\mathbf{S}$.

As $X \in \mathbb{X}$ then $B \subseteq X \subseteq A \cup B$ by the construction. Therefore $X = B + (A - A')$ for some $A' \subseteq A$. We show first that for all $a \in A$, $a \sim \Sigma X$. If $a \in X$ then $a \sim \Sigma X$ by the definition. If not, then put $Y = X + \{a\}$. Then $X \neq Y$, $X \subseteq Y$, $B \subseteq Y \subseteq A \cup B$ and Y is dependent in $\mathbf{S}$.

By the definition there exist elements: $\{x_0, x_1, ..., x_n\} \in \Delta$ with $x_1, ..., x_n \in X + \{a\}$, as X is an independent set. Moreover, one of x_i is a, say $x_0 = a$. We get $a \sim \Sigma X$ as $x_0, x_1, ..., x_n$ are different.

Now we show that X is a basis of $\mathbf{S}$. Let $x \in S$, then $x \sim \Sigma A$ as A is a basis of $\mathbf{S}$. Moreover for all $a \in A$, $a \sim \Sigma X$, thus $x \sim \Sigma X$ by the Transitivity Axiom. □

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