Brandt λ^0-extensions of monoids with zero

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Let S be a semigroup with zero and I_λ be a set of cardinality $\lambda \geqslant 1$. We define the semigroup operation on the set $B_\lambda(S) = (I_\lambda \times S \times I_\lambda) \cup \{0\}$ as follows:

$$(\alpha, a, \beta) \cdot (\gamma, b, \delta) = \begin{cases} (\alpha, ab, \delta), & \text{if } \beta = \gamma; \\ 0, & \text{if } \beta \neq \gamma, \end{cases}$$

and $(\alpha, a, \beta) \cdot 0 = 0 \cdot (\alpha, a, \beta) = 0 \cdot 0 = 0$, for all $\alpha, \beta, \gamma, \delta \in I_\lambda$ and $a, b \in S$. Obviously, if S has zero then $\mathcal{J} = \{0\} \cup \{(\alpha, 0_S, \beta) \mid 0_S \text{ is the zero of } S\}$ is an ideal of $B_\lambda(S)$. We put $B^0_\lambda(S) = B_\lambda(S)/\mathcal{J}$ and the semigroup $B^0_\lambda(S)$ is called the *Brandt λ^0-extension of the semigroup S with zero* [1]. Next, if $A \subseteq S$ then we shall denote $A_{\alpha\beta} = \{(\alpha, s, \beta) \mid s \in A\}$ if A does not contain zero, and $A_{\alpha,\beta} = \{(\alpha, s, \beta) \mid s \in A \setminus \{0\}\} \cup \{0\}$ if $0 \in A$, for $\alpha, \beta \in I_\lambda$.

Proposition 1. *Let $\lambda \geqslant 1$ be any cardinal. Then the Brandt λ^0-extension of a semigroup S is a regular, orthodox, inverse, 0-simple, completely 0-simple, primitive inverse or congruence-free semigroup if and only if such is also S.*

Proposition 2. *Let $\lambda_1, \lambda_2 \geqslant 1$ be any cardinals, S be a semigroup and S_1 be the Brandt λ_1^0-extension of the semigroup S. Then the Brandt λ_2^0-extension of the semigroup S_1 is isomorphic to the Brandt λ^0-extension of the semigroup S for the cardinal $\lambda = \lambda_1 \cdot \lambda_2$.*

Proposition 3. *Let S be a monoid with zero, $\lambda \geqslant 1$ be any cardinal, and $B^0_\lambda(S)$ be the Brandt λ^0-extension of S. Then every non-trivial homomorphic image of $B^0_\lambda(S)$ is the Brandt λ^0-extension of some monoid with zero. Moreover, if T is the image of $B^0_\lambda(S)$ under a homomorphism h, then T is isomorphic to the Brandt λ^0-extension of the homomorphic image of the monoid $S_{\alpha,\alpha}$ under the homomorphism h for any $\alpha \in I_\lambda$.*

Definition 4. Let λ be any cardinal $\geqslant 2$. We shall say that a semigroup S has the *$\mathcal{B}^*$-property* if S does not contain the semigroup of 2×2-matrix units and that S has the *$\mathcal{B}^*_\lambda$-property* if S satisfies the following conditions: S does not contain the semigroup of $I_\lambda \times I_\lambda$-matrix units and S does not contain the semigroup of 2×2-matrix units B_2 such that the zero of B_2 is the zero of T.

Theorem 5. *Let λ_1 and λ_2 be cardinals such that $\lambda_2 \geqslant \lambda_1 \geqslant 1$. Let $B^0_{\lambda_1}(S)$ and $B^0_{\lambda_2}(T)$ be the Brandt λ_1^0- and λ_2^0-extensions of monoids S and T with zero, respectively. Let $h\colon S \to T$ be a homomorphism such that $(0_S)h = 0_T$ and suppose that $\varphi\colon I_{\lambda_1} \to I_{\lambda_2}$ is a one-to-one map. Let e be a non-zero idempotent of T, H_e a maximal subgroup of T with the unity e and $u\colon I_{\lambda_1} \to H_e$ a map. Then*

$I_h = \{s \in S \mid (s)h = 0_T\}$ is an ideal of S and the map $\sigma\colon B^0_{\lambda_1}(S) \to B^0_{\lambda_2}(T)$ defined by the formulae

$$((\alpha, s, \beta))\sigma = \begin{cases} ((\alpha)\varphi, (\alpha)u \cdot (s)h \cdot ((\beta)u)^{-1}, (\beta)\varphi), & \text{if } s \notin S \setminus I_h; \\ 0_2, & \text{if } s \in I_h^*, \end{cases}$$

and $(0_1)\sigma = 0_2$ is a non-trivial homomorphism from $B^0_{\lambda_1}(S)$ into $B^0_{\lambda_2}(T)$. Moreover, if for the semigroup T the following assertions hold:

(i) every idempotent of T lies in the center of T; and

(ii) T has the $\mathcal{B}^*_{\lambda_1}$-property,

then every non-trivial homomorphism from $B^0_{\lambda_1}(S)$ into $B^0_{\lambda_2}(T)$ can be constructed in this manner.

Let S and T be monoids with zeros. Let $\mathtt{Hom}_0(S,T)$ be the set of all homomorphisms $\sigma\colon S \to T$ such that $(0_S)\sigma = 0_T$. We put

$$\mathbf{E}_1(S,T) = \{e \in E(T) \mid \text{there exists } \sigma \in \mathtt{Hom}_0(S,T) \text{ such that } (1_S)\sigma = e\}$$

and define the family

$$\mathscr{H}_1(S,T) = \{H(e) \mid e \in \mathbf{E}_1(S,T)\},$$

where we denote the maximal subgroup with the unity e in the semigroup T by $H(e)$. Also by $\mathfrak{B}$ we denote the class of monoids S with zeros such that S has $\mathcal{B}^*$-property and every idempotent of S lies in the center of S.

We define a category $\mathscr{B}$ as follows: $\mathbf{Ob}(\mathscr{B}) = \{(S, I) \mid S \in \mathfrak{B}$ and I is a non-empty set$\}$, and if S is a trivial semigroup then we identify (S, I) and (S, J) for all non-empty sets I and J; and $\mathbf{Mor}(\mathscr{B})$ consists of triples $(h, u, \varphi)\colon (S, I) \to (S', I')$, where

$$\begin{aligned} &h\colon S \to S' \text{ is a homomorphism such that } h \in \mathtt{Hom}_0(S, S'), \\ &u\colon I \to H(e) \text{ is a map, for } H(e) \in \mathscr{H}_1(S, S'), \qquad (1) \\ &\varphi\colon I \to I' \text{ is an one-to-one map,} \end{aligned}$$

with the composition

$$(h, u, \varphi)(h', u', \varphi') = (hh', [u, \varphi, h', u'], \varphi\varphi'),$$

where the map $[u, \varphi, h', u']\colon I \to H(e)$ is defined by the formula

$$(\alpha)[u, \varphi, h', u'] = ((\alpha)\varphi)u' \cdot ((\alpha)u)h' \qquad \text{for } \alpha \in I.$$

A straightforward verification shows that $\mathscr{B}$ is a category with the identity morphism $\varepsilon_{(S,I)} = (\mathrm{Id}_S, u_0, \mathrm{Id}_I)$ for any $(S, I) \in \mathbf{Ob}(\mathscr{B})$, where $\mathrm{Id}_S\colon S \to S$ and $\mathrm{Id}_I\colon I \to I$ are identity maps and $(\alpha)u_0 = 1_S$ for all $\alpha \in I$.

We define the category $\mathscr{B}^*(\mathscr{S})$ as follows: $\mathbf{Ob}(\mathscr{B}^*(\mathscr{S}))$ are all Brandt λ^0-extensions of monoids S with zeros such that S has the $\mathcal{B}^*$-property and every idempotent of S lies in the center of S; and $\mathbf{Mor}(\mathscr{B}^*(\mathscr{S}))$ are homomorphisms of the Brandt λ^0-extensions of monoids S with zeros such that S has the $\mathcal{B}^*$-property and every idempotent of S lies in the center of S.

For each $(S, I_{\lambda_1}) \in \mathbf{Ob}(\mathscr{B})$ with non-trivial S, let $\mathbf{B}(S, I_{\lambda_1}) = B^0_{\lambda_1}(S)$ be the Brandt λ^0-extension of the semigroup S. For each $(h, u, \varphi) \in \mathbf{Mor}(\mathscr{B})$ with a non-trivial homomorphism h, where $(h, u, \varphi)\colon (S, I_{\lambda_1}) \to (T, I_{\lambda_2})$ and $(T, I_{\lambda_2}) \in$

$\mathbf{Ob}(\mathscr{B})$, we define a map $\mathbf{B}(h,u,\varphi)\colon \mathbf{B}(S,I_{\lambda_1}) = B^0_{\lambda_1}(S) \to \mathbf{B}(T,I_{\lambda_2}) = B^0_{\lambda_2}(T)$ as follows:

$$((\alpha,s,\beta))[\mathbf{B}(h,u,\varphi)]=\begin{cases} ((\alpha)\varphi, (\alpha)u\cdot(s)h\cdot((\beta)u)^{-1}, (\beta)\varphi), & \text{if } s \notin S\setminus I_h;\\ 0_2, & \text{if } s\in I_h^*,\end{cases}$$

and $(0_1)[\mathbf{B}(h,u,\varphi)] = 0_2$, where $I_h = \{s \in S \mid (s)h = 0_T\}$ is an ideal of S and 0_1 and 0_2 are the zeros of the semigroups $B^0_{\lambda_1}(S)$ and $B^0_{\lambda_2}(T)$, respectively. For each $(h,u,\varphi)\in \mathbf{Mor}(\mathscr{B})$ with a trivial homomorphism h we define a map $\mathbf{B}(h,u,\varphi)\colon \mathbf{B}(S,I_{\lambda_1}) = B^0_{\lambda_1}(S) \to \mathbf{B}(T,I_{\lambda_2}) = B^0_{\lambda_2}(T)$ as follows: $(a)[\mathbf{B}(h,u,\varphi)] = 0_2$ for all $a \in \mathbf{B}(S,I_{\lambda_1}) = B^0_{\lambda_1}(S)$. If S is a trivial semigroup then we define $\mathbf{B}(S,I_{\lambda_1})$ to be a trivial semigroup.

A functor $\mathbf{F}$ from a category $\mathscr{C}$ into a category $\mathscr{K}$ is called *full* if for any $a,b\in\mathbf{Ob}(\mathscr{C})$ and for any $\mathscr{K}$-morphism $\alpha\colon \mathbf{F}a\to\mathbf{F}b$ there exists a $\mathscr{C}$-morphism $\beta\colon a\to b$ such that $\mathbf{F}\beta=\alpha$, and $\mathbf{F}$ called *representative* if for any $a\in\mathbf{Ob}(\mathscr{K})$ there exists $b\in\mathbf{Ob}(\mathscr{C})$ such that a and $\mathbf{F}b$ are isomorphic.

Theorem 6. *$\boldsymbol{B}$ is a full representative functor from $\mathscr{B}$ into $\mathscr{B}^*(\mathscr{S})$.*

We observe that the statements similar to Theorem 6 hold for the categories of inverse and Clifford inverse semigroups.

We define the category $\mathscr{B}\mathscr{S}\mathscr{L}$ as follows: $\mathbf{Ob}(\mathscr{B}\mathscr{S}\mathscr{L}) = \{(S,I) \mid S$ is a semilattice with unity and zero and I is a non-empty set$\}$, and if S is a trivial semigroup then we identify (S,I) and (S,J) for all non-empty sets I and J and $\mathbf{Mor}(\mathscr{B}\mathscr{S}\mathscr{L})$ consists of corresponding triples $(h,u,\varphi)\colon\ (S,I)\to(S',I')$, which satisfy condition (1).

The second category $\mathscr{B}^*(\mathscr{S}\mathscr{L})$ is defined as follows: $\mathbf{Ob}(\mathscr{B}^*(\mathscr{S}\mathscr{L}))$ are all Brandt λ^0-extensions of semilattices with zeros and identities and $\mathbf{Mor}(\mathscr{B}^*(\mathscr{S}\mathscr{L}))$ are homomorphisms of the Brandt λ^0-extensions of semilattices with zeros and identities.

Theorem 7. *The categories $\mathscr{B}\mathscr{S}\mathscr{L}$ and $\mathscr{B}^*(\mathscr{S}\mathscr{L})$ are isomorphic.*

Also we describe the structure of compact and countably compact topological Brandt λ^0-extensions of topological monoids with zero and show that categories of compact and countably compact topological Brandt λ^0-extensions of topological monoids with zero have similar property to Brandt λ^0-extensions of monoids with zero.

References

[1] Gutik O. V., Pavlyk K. P.: *On Brandt λ^0-extensions of semigroups with zero*, Mat. Metody Phis.-Mech. Polya. **49**:3 (2006), 26–40.