On mode reducts of monoids

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We characterize mode reducts of commutative monoids. We show that they are equivalent to certain ternary algebras coming from the regularization of the variety of integral affine spaces, and to some n-ary semilattices. For modes theory and related topics see [7], [8] and [9].

1. Some varieties of commutative monoids

The lattice $L(\mathcal{CM})$ of varieties of commutative monoids $(M, +, 0)$ is isomorphic to the lattice of congruence relations of the monoid $(\mathbb{N}, +, 0)$. See [3] and [4]. Each non-trivial subvariety of the variety $\mathcal{CM}$ of commutative monoids is defined by one additional identity

$$(m+n)x = mx, \tag{1}$$

where $m \in \mathbb{N}$ and $n \in \mathbb{Z}^+$. Such variety is denoted by $\mathcal{C}_{m+n,m}$. For a fixed m, the varieties $\mathcal{C}_{m+n,m}$ form a sublattice isomorphic to the lattice of positive natural numbers with divisibility as an ordering relation. Each variety $\mathcal{C}_{n,0}$ is equivalent to the variety $\mathcal{AG}_n$ of Abelian groups satisfying the identity $nx = 0$. Also each variety $\mathcal{AG}_n$ is equivalent to the variety $\mathcal{MOD}_n$ of modules over the ring $\mathbb{Z}_n$.

Any variety $\mathcal{C}_{n,0}$ may be defined by the three identities defining commutative monoids and by $x + ny = x$. We apply Płonka's theory of regularized varieties of algebras with constant operations(see e.g. [5], [6] and [9]) to describe the structure of monoids in the regularization $\tilde{\mathcal{C}}_{n,0}$ of $\mathcal{C}_{n,0}$.

Proposition 1. *The following classes coincide:*
(a) the variety $\mathcal{C}_{1+n,1}$*;*
(b) the regularization $\tilde{\mathcal{C}}_{n,0}$ *of* $\mathcal{C}_{n,0}$*;*
(c) the class of Płonka sums of monoids in $\mathcal{C}_{n,0}$*.*

Monoids in the varieties $\mathcal{C}_{1+n,1}$ also have the structure of semimodules over commutative semirings with identity. The semiring is isomorphic to the semiring $\mathbb{N}_{1+n,1}$ obtained as the quotients $\mathbb{N}/cg(1+n, 1)$ of the semiring $\mathbb{N}$ of natural numbers by the principal congruence generated by $(1 + n, 1)$. Each variety $\mathcal{C}_{1+n,1}$ is equivalent to the variety $\mathcal{SMOD}_{1+n,1}$ of semimodules over the semiring $\mathbb{N}_{1+n,1}$. Both the semiring $\mathbb{N}_{1+n,1}$ and the variety $\mathcal{SMOD}_{1+n,1}$ satisfy the identity $x(1 + n) = x$.

2. Mode reducts of commutative monoids

By associativity and commutativity of monoid addition, also all derived monoid operations satisfy entropic laws. It follows that to find mode reducts of commutative monoids it is sufficient to look for its idempotent reducts.

Full idempotent reducts of $\mathbb{Z}$-modules $(A, +, \mathbb{Z})$ are given by integral affine spaces, reducts $(A, P, \underline{\mathbb{Z}})$ of $(A, +, \mathbb{Z})$, where $xyzP = x - y + z$ is the ternary Mal'cev operation, and for each $n \in \mathbb{Z}$, there is a binary operation $xy\underline{n} = x(1-n) + yn$. (See [9].) The reduct (A, P) and affine space $(A, P, \underline{\mathbb{Z}})$ are equivalent. The class of all such algebras forms a variety. It is denoted by $\underline{\underline{\mathbb{Z}}}$.

The lattice $L(\underline{\underline{\mathbb{Z}}})$ of subvarieties of $\underline{\underline{\mathbb{Z}}}$ is isomorphic to the lattice $L(\mathcal{AG})$ of varieties of abelian groups. Under this isomorphism one assigns to each subvariety $\mathcal{AG}_n$ of $\mathcal{AG}$ the subvariety $\underline{\underline{\mathbb{Z}}}_n$ of $\underline{\underline{\mathbb{Z}}}$ of affine $\mathbb{Z}_n$-spaces, defined by the additional identity $y(x(y(\dots)yP)xP)yP = x$, where y is repeated n times.

We would like to find full idempotent reducts of commutative monoids.

Proposition 2. *Each idempotent derived (non-trivial) operation of a monoid in the variety $\mathcal{C}_{1+n,1}$ may be obtained from the operation*

$$\omega_{n+1}(x_1, \dots, x_{n+1}) = x_1 + x_2 + \dots x_{n+1}$$

by a suitable compositions of ω_{n+1} and identification of variables.

Let us note that in the variety $\mathcal{C}_{n,0}$, the Mal'cev operation P can be written as $xyzP = x + (n-1)y + z$. It is also a term operation of $\mathcal{C}_{1+n,1}$-monoids, though in $\mathcal{C}_{1+n,1}$, it is no more a Mal'cev operation.

Theorem 3. *Let $(M, +, 0)$ be a monoid in the variety $\mathcal{C}_{1+n,1}$. Then the following three sets of derived operations coinside:*

(a) the set of idempotent derived operations of $(M, +, 0)$,
(b) the set of derived operations of the reduct (M, P),
(c) the set of derived operations of the reduct (M, ω_{n+1}).

3. Semiaffine $\mathbb{N}_{1+n,1}$-spaces and $(n+1)$-semilattices

By analogy with affine spaces, we will call the full idempotent reducts of $\mathcal{C}_{1+n,1}$-monoids, *semiaffine spaces* over $\mathbb{N}_{1+n,1}$ or *semiaffine* $\mathbb{N}_{1+n,1}$*-spaces.* Results of Section 2 show that the semiaffine space that is a reduct of a $\mathcal{C}_{1+n,1}$-monoid M is equivalent to the reducts (M, P) and (M, ω_{n+1}).

For an idempotent irregular variety $\mathcal{V}$ without constant operations, its regularization $\widetilde{\mathcal{V}}$ coincides with the class of Płonka sums of $\mathcal{V}$-algebras over arbitrary semilattice.

Let us call Płonka sums of $\mathcal{V}$-algebras over semilattices with the smallest element *bounded Płonka sums* and denote by $\widetilde{\mathcal{V}}^b$ the class of all such algebras.

Theorem 4. *The subclass $\tilde{\underline{\underline{\mathbb{Z}}}}_n^b$ of bounded Płonka sums of the regularization $\tilde{\underline{\underline{\mathbb{Z}}}}_n$ of the variety $\underline{\underline{\mathbb{Z}}}_n$ consists precisely of algebras term equivalent to full idempotent reducts of $\mathcal{C}_{1+n,1}$-monoids.*

The full idempotent reducts of $\mathcal{C}_{1+n,1}$-monoids may also be described using the operation ω_{n+1}. This operation is idempotent and satisfies certain generalized commutativity and associativity. Such algebras with one $(n+1)$-ary operation are called $(n+1)$-*semilattice*. The $(n+1)$-semilattices are modes. Hence the reduct (M, ω_{n+1}) of any $\mathcal{C}_{1+n,1}$-monoid $(M, +, 0)$ is an $(n+1)$-semilattice.

Let $\mathcal{SL}_{n+1}$ be the variety of $(n+1)$-semilattices. And let $\mathcal{SL}^0_{n+1}$ be its subvariety defined by the identity $\omega_{n+1}(x, y, \ldots, y) = x$.

Lemma 5. *(a) The variety $\mathcal{SL}_{n+1}$ of $(n+1)$-semilattices is the regularization of the subvariety $\mathcal{SL}^0_{n+1}$.*

(b) Let (A, ω_{n+1}) be a member of $\mathcal{SL}^0_{n+1}$. Define a ternary operation $xyzP$ on A by

$$xyzP := \omega_{n+1}(x, y, \ldots, y, z).$$

Then (A, P) is a member of the variety $\underline{\underline{\mathbb{Z}}}_n$.

(c) Let (A, P) be an algebra in $\underline{\underline{\mathbb{Z}}}_n$. Define an $(n+1)$-ary operation $\omega_{n+1}(x_1, x_2, \ldots, x_{n+1})$ on A by:

$$\omega_{n+1}(x_1, x_2, \ldots, x_{n+1}) = (\ldots((x_1x_2x_3P)x_2x_4P)\ldots)x_2x_{n+1}P.$$

Then (A, ω_{n+1}) is an $(n+1)$-semilattice in the variety $\mathcal{SL}^0_{n+1}$.

(d) The varieties $\mathcal{SL}^0_{n+1}$ and $\underline{\underline{\mathbb{Z}}}_n$ are equivalent.

Theorem 6. *The varieties $\mathcal{SL}_{n+1}$ and $\widetilde{\underline{\underline{\mathbb{Z}}}}_n$ are equivalent.*

Theorem 7. *The subclass $\mathcal{SL}^b_{n+1}$ of bounded Płonka sums of the regularization $\mathcal{SL}_{n+1}$ of the variety $\mathcal{SL}^0_{n+1}$ consists precisely of algebras term equivalent to full idempotent reducts of $\mathcal{C}_{1+n,1}$-monoids.*

4. Semiaffine $\mathbb{Z}$-spaces

The full idempotent reducts of $\mathcal{C}_{1+n,1}$-monoids form a subclass of the regularization $\widetilde{\underline{\underline{\mathbb{Z}}}}_n$ of the variety $\underline{\underline{\mathbb{Z}}}_n$. In contrast with the varieties $\mathcal{AG}_n$, the variety $\mathcal{AG}$ of all abelian groups is not (equivalent to) a variety of commutative monoids. To represent algebras in the regularization $\widetilde{\mathcal{AG}}$ of $\mathcal{AG}$, in a similar way as in the case of regularization of $\mathcal{AG}_n$, we have to use commutative Clifford monoids. *Clifford monoid* is a commutative monoid with a unary operation $-$ satisfying the identities: $x+(-x)+x = x$, $-(-x)=x$ and $-(x+y)=(-x)+(-y)$.

The varieties $\widetilde{\mathcal{AG}}$ and the variety of commutative Clifford monoids coincide. We will call the full idempotent reducts of commutative Clifford monoids *semiaffine $\mathbb{Z}$-spaces.*

Proposition 8. *Each (non-trivial) idempotent derived operation of a commutative Clifford monoid $(A, +, -, 0)$ may be obtained from the regMal'cev operation P by a suitable composition of P and identification of variables.*

As in the case of Theorem 4, one obtains the following theorem.

Theorem 9. *The subclass $\widetilde{\underline{\underline{\mathbb{Z}}}}^b$ of bounded Płonka sums of the regularization $\widetilde{\underline{\underline{\mathbb{Z}}}}$ of the variety $\underline{\underline{\mathbb{Z}}}$ consists precisely of algebras term-equivalent to semiaffine $\mathbb{Z}$-spaces.*

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