Monounary algebras and their pseudovarieties[1]

Danica Jakubíková-Studenovská
e-mail: danica.studenovska@upjs.sk
P. J. Šafárik University, Košice, Slovakia

A variety of algebras is defined as the system of those algebras which satisfy a given set of identities.

Monounary algebras are algebras with one unary operation. Beyond doubt, they have a considerable and significant role in the study of algebraic and relational structures (cf., e.g., B. Jónsson [5], L. A. Skornjakov [7], J. Chvalina [2], M. Novotný [6]).

According to [4], each variety of monounary algebras is determined by one identity. These varieties are of the form

- the class $\mathcal{U}$ of all monounary algebras,
- the class $\mathcal{V}_{nk}$ determined by the identity $f^{n+k}(x) \approx f^k(x)$, $n \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$,
- the class $\mathcal{V}_k$ determined by the identity $f^{k+1}(x) \approx f^k(y)$, $k \in \mathbb{N} \cup \{0\}$.

It is known that each variety can be characterized as a class which is closed with respect to the operators **H** (forming homomorphic images), **S** (forming subalgebras), **P** (forming direct products). For a class of finite algebras it turned out to be useful to work with the operators **H**, **S** and $\mathbf{P}_f$, where $\mathbf{P}_f$ means forming products of finitely many algebras. A system of finite algebras is said to be a *pseudovariety* if it is closed with respect to the operators **H**, **S** and $\mathbf{P}_f$.

Similarly as an analogous result for varieties, the following result is valid:

The smallest pseudovariety containing a class $\mathcal{K}$ of algebras of a given type is equal to $\mathbf{HSP}_f\mathcal{K}$; this class is called a *pseudovariety generated by* $\mathcal{K}$.

Pseudovarieties of algebras were investigated by several authors; for more references c.f., e.g., the monograph of J. Almeida [1]. S.Eilenberg and M.P.Schützenberger [3] proved that if $\mathcal{P}$ is a pseudovariety of algebras of a f inite type, then there is a sequence ε_n, $n \geq 1$ of identities such that

$$\mathcal{P} = \bigcup_{k \geq 1} \mathcal{P}_k,$$

where $\mathcal{P}_k$ is the family of all finite algebras satisfying the system of identities $\{\varepsilon_n : n \geq k\}$.

Our aim is to describe all pseudovarieties of monounary algebras. Namely, we can give a constructive (using finite products, homomorphisms and subalgebras of a simple set of generators) description of members of all pseudovarieties of monounary algebras.

A pseudovariety $\mathcal{P}$ is called *equational*, if there is a variety $\mathcal{V}$ such that $\mathcal{P}$ consists of all finite members of $\mathcal{V}$. A pseudovariety $\mathcal{P}$ is said to be *finitely generated*, if there exists a finite set of algebras generating $\mathcal{P}$.

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In view of [1], each finitely generated pseudovariety is equational. We prove the "converse" (for more detail see below): each equational pseudovariety of monounary algebras except the pseudovariety of all finite monounary algebras, is finitely gen erated. Also, it will be shown that each equational pseudovariety of monounary algebras except the pseudovariety of all finite monounary algebras can be generated by a single algebra.

Let $k \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$. We denote by (A_{nk}, f) a connected monounary algebra generated by one element, such that (A_{nk}, f) contains an $n-$element cycle and $\operatorname{card} A_{nk} = n + k$. Next, (A'_{1k}, f) is a disjoint union of (A_{1k}, f) and a 1-element cycle.

Theorem 1. *A class $\mathcal{P}$ of finite monounary algebras is a finitely generated pseudovariety if and only if one of the following conditions is satisfied:*

(a) *There are $k \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$ such that $\mathcal{P}$ consists of all finite monounary algebras (A, f) with the property that if B is a connected component of (A, f), then*
  (i) *(B, f) has a cycle of length dividing n,*
  (ii) *(B, f) is of depth at most k.*

(b) *There is $k \in \mathbb{N} \cup \{0\}$ such that $\mathcal{P}$ consists of all finite monounary algebras (A, f) with the following properties:*
  (i) *(A, f) is connected,*
  (ii) *(A, f) possesses a 1-element cycle,*
  (iii) *(A, f) is of depth at most k.*

In the case (a)*, if $n > 1$, then $\mathcal{P}$ is generated by the algebra (A_{nk}, f) and if $n = 1$, then by (A'_{1k}, f).*

In the case (b)*, $\mathcal{P}$ is generated by the algebra (A_{1k}, f).*

This theorem implies the above mentioned result:

Theorem 2. *A class $\mathcal{P}$ of finite monounary algebras is a finitely generated pseudovariety if and only if $\mathcal{P}$ consists of all finite algebras belonging to some variety $\mathcal{V}$ of monounary algebras such that $\mathcal{V}$ fails to be equal to the class of all monounary algebras. Cons equently, if $\mathcal{P}$ is a pseudovariety of monounary algebras and it is not equal to the class of all monounary algebras, then $\mathcal{P}$ is finitely generated if and only if $\mathcal{P}$ is equational.*

Now let us consider a general case: suppose that $\mathcal{P}$ is a pseudovariety. Each algebra of $\mathcal{P}$ can be generated by algebras of types (A_{nk}, f) or (A'_{1k}, f), hence also $\mathcal{P}$ can be generated by algebras of these types; deno te them S_m, $m \in \mathbb{N}$. For $n \in \mathbb{N}$ let $\mathcal{P}_n$ be a (finitely generated) pseudovariety, which is generated by the set $\{S_m : m \leq n\}$. Then

$$\mathcal{P} = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n, \tag{1}$$

i.e., the pseudovariety $\mathcal{P}$ is a union of an increasing countable chain of finitely generated pseudovarieties.

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