Subalgebras of the square of minimal majority algebra

Kalle Kaarli
e-mail: kaarli@ut.ee
University of Tartu, Estonia

An algebra $\mathbf{A}$ having a majority term will be called *a majority algebra.* If $\mathbf{A}$ is any algebra then the set $S_2(\mathbf{A})$ of all subuniverses of $\mathbf{A} \times \mathbf{A}$ has a natural algebraic structure: the binary operations $\circ$ (relational product) and $\cap$, the unary operation $\breve{}$ (taking inverse relation) and the nullary operations Δ (diagonal) and $\nabla = A \times A$. C. Bergman denoted in [1] this structure by $\mathbf{S}_2(\mathbf{A})$. He proved that the categorical equivalence class of any finite majority algebra $\mathbf{A}$ is determined by $\mathbf{S}_2(\mathbf{A})$. For technical reasons he assumed that $\mathbf{A}$ had no nullary fundamental operations and thus in his approach $S_2(\mathbf{A})$ always contained $\emptyset$. This restriction was not a principal one because one can always replace nullary operations by unary constant operations.

We prefer a slightly different approach. Our $S_2(\mathbf{A})$ is a set of all subalgebras of $\mathbf{A} \times \mathbf{A}$, so it never contains $\emptyset$. Since we do not require the existence of nullary fundamental operations, now the intersection is not an algebraic operation on $S_2(\mathbf{A})$, in general. Therefore in our approach $\mathbf{S}_2(\mathbf{A})$ is the ordered structure $\langle S_2(\mathbf{A}); \circ, \breve{}, \Delta, \nabla, \subseteq\rangle$.

C. Bergman's basic result in [1] was that two finite majority algebras $\mathbf{A}$ and $\mathbf{B}$ are categorically equivalent iff the algebraic structures $\mathbf{S}_2(\mathbf{A})$ and $\mathbf{S}_2(\mathbf{B})$ are isomorphic. The latter is clearly equivalent to the isomorphism of our ordered structures.

J. Snow posed in [5] the problem: characterize the structures that appear as $\mathbf{S}_2(\mathbf{A})$ for some finite majority algebra. Here we solve the problem for *minimal* algebras, that is, for algebras with no proper subalgebras.

Clearly the structures $\mathbf{S}$ we are interested in must be ordered monoids with involution and zero element. Thus, $\mathbf{S} = \langle S; \cdot, ^{-1}, 1, 0, \leq\rangle$ and the following conditions must be satisfied:

- $\langle S; \cdot, 1, 0\rangle$ is a monoid with zero;
- $(xy)^{-1} = y^{-1}x^{-1}$ for every $x, y \in S$;
- $(x^{-1})^{-1} = x$ for every $x \in S$;
- $\leq$ is a partial order relation;
- for every $x, y, u, v \in S$, if $x \leq y$ and $u \leq v$ then $xu \leq yv$;
- for every $x, y \in S$, if $x \leq y$ then $x^{-1} \leq y^{-1}$.

It is clear that the operation symbols $\cdot, ^{-1}, 1, 0$ correspond in $\mathbf{S}_2(\mathbf{A})$ to $\circ, \breve{}, \Delta, \nabla$, respectively. The relation symbol $\leq$, however, corresponds to $\supseteq$, not to $\subseteq$, as one might think. This seemingly strange agreement has several reasons. In particular, it agrees with standard definition of the natural order in inverse semigroups. This immediately implies that our structure $\mathbf{S}$ is a meet semilattice but not a join semilattice, in general. Our main result is the following (the notions *E-reflexivity, completeness, distributivity*) have been taken from the theory of inverse monoids.

Theorem 1. *Let* **S** *be a meet semilattice ordered monoid with involution and the zero element. Then the following are equivalent:*

(1) there exists a finite minimal majority algebra **A** *such that* **S** *is isomorphic to* $\mathbf{S}_2(\mathbf{A})$*;*

(2) **S** *satisfies the following conditions:*

(a) (E-reflexivity) for every $x, y \in S$*, if* $x^{-1}y \leq 1$ *then* $xy^{-1} \leq 1$*;*

(b) (completeness) for every $x, y \in S$*, the join* $x \vee y$ *exists if and only if* $x^{-1}y \leq 1$*;*

(c) (distributivity) for every $x, y, z \in S$*, if* $y \vee z$ *exists then also* $xy \vee xz$ *exists and the equality* $x(y \vee z) = xy \vee xz$ *holds.*

There are two important special cases. First consider the case when $\langle S; \cdot, ^{-1}, 1\rangle$ is an inverse monoid. Then it is easy to check that the order relation $\leq$ of **S** is precisely the natural order of that inverse monoid. It has been proved in [3] and [4] that finite, E-reflexive, complete, distributive, inverse monoids with zero are precisely (up to isomorphism) the structures $\mathbf{S}_2(\mathbf{A})$ for finite minimal algebras **A** generating arithmetical varieties. Now this result easily follows from Theorem 1.

The other important special case is given by condition $x \leq 1$ for every $x \in S$. In terms of algebra **A** this means that every subalgebra of $\mathbf{A}^2$ contains Δ. Note that in presence of majority term the latter is equivalent to the condition that all constants are terms. In this case $\langle S; \leq\rangle$ turns out to be a lattice and the whole structure **S** becomes a *lattice ordered semiring* (cf. [2]). Thus, lattice ordered semirings with involution are precisely (up to isomorphism) the structures $\mathbf{S}_2(\mathbf{A})$ where **A** is a finite majority algebra with all constants being terms.

References

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