A general Galois theory for dual operations and dual relations

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In the talk, we will introduce a Galois connection between dual operations and what we will call dual relations on an arbitrary object in a concrete category. In analogy to the well-known Galois connection Pol-Inv, the Galois closed sets of our introduced Galois connection are characterized by local closures of clones of dual operations and of what we will introduce as clones of dual relations.

1. Motivation and setting.

It is a well-known result that clones of operations can externally be characterized as Galois closed sets with respect to the Galois connection Pol-Inv between operations and relations based on the property of an operation to preserve a relation (for more details, we refer to [3] and [4]).

In [5], a similar Galois theory is developed for cofunctions and what the authors call corelations.

Definition 1. For a (possibly infinite) set A, an n-ary *cofunction* (or *co-operation*) on A is mapping from A to $\{1, ..., n\} \times A$. For a set F of cofunctions, denote by $F^{(n)}$ the set of n-ary operations in F.

Note that $\{1, ..., n\} \times A$ can be interpreted as the n-th copower of A.

Definition 2 ([5])**.** For a set A, an n-ary *corelation* on A is a subset of $\{1, ..., n\}^A$ (i.e. it is a set of mappings from A to $\{1, ..., n\}$).

Definition 3 ([5])**.** Let f be an n-ary cofunction and let σ be a corelation on a set A. Say that f *preserves* σ, written $f \rhd \sigma$, if, for all $r_1, ..., r_n \in \sigma$, we have $[r_1, ..., r_n] \circ f \in \sigma$, where $[r_1, ..., r_n]$ denotes the co-tupling of $r_1, ..., r_n$.

Definition 4 ([5])**.** Let F be a set of cofunctions on A and let R be a set of corelations on A. Define

$$\begin{aligned}\operatorname{cInv} F &:= \{\sigma \text{ corelation on } A \mid \forall f \in F : f \rhd \sigma\},\\ \operatorname{cPol} R &:= \{f \text{ cofunction on } A \mid \forall \sigma \in R : f \rhd \sigma\}.\end{aligned}$$

It is shown by the authors that, analogue to Pol-Inv, the Galois closed sets can be characterized nicely: they are local closures of clones of cofunctions as introduced in [1] and clones of corelations as introduced in the paper itself.

Definition 5 ([1])**.** A set of cofunctions F is a *coclone* (or *clone of cofunctions*) if F contains all injections $\iota_i^n : A \to \{1, ..., n\} \times A, \iota_i^n(a) := (i, a)$ ($n \in \mathbb{N}$, $i \in \{1, ..., n\}$) and, for $f \in F^{(n)}$, $g_1, ..., g_n \in F^{(k)}$, the superposition $f \cdot [g_1, ..., g_n] := [g_1, ..., g_n] \circ f$ is also in F.

In order to pave the way for a generalization, note that cofunctions are a special case of so called dual operations.

Definition 6. Let $\mathcal{X}$ be a category with finite, nonempty coproducts and let $\mathbf{X} \in \mathcal{X}$. An n-ary dual operation over $\mathbf{X}$ is a morphism from $\mathbf{X}$ to the n-th copower of $\mathbf{X}$. Denote the set of all n-ary dual operations over $\mathbf{X}$ by $\overline{O}_{\mathbf{X}}^{(n)}$ and set $\overline{O}_{\mathbf{X}} := \bigcup_{n\geq 1} \overline{O}_{\mathbf{X}}^{(n)}$.

The notion of dual operations generalizes the notion of cofunctions, since, in the category of sets, the dual operations on a set X are precisely the cofunctions on X if we identify the n-th copower of X with $\{1, ..., n\} \times X$. We can also generalize the notion of coclones.

Definition 7. Call a set F of dual operations over $\mathbf{X}$ a *clone of dual operations over* $\boldsymbol{X}$, written $F \leq \overline{O}_{\mathbf{X}}$, if F contains all the injection morphisms from $\mathbf{X}$ to finite copowers of $\mathbf{X}$ and, for $f \in F^{(n)}$, $g_1, ..., g_n \in F^{(k)}$, the superposition $f \cdot [g_1, ..., g_n] := [g_1, ..., g_n] \circ f$ is also in F. All clones of dual operations over $\mathbf{X}$ form a lattice with respect to inclusion. Denote this lattice by $\overline{\mathcal{L}}_{\mathbf{X}}$.

It was shown in [2] that, for every finite centralizer clone C (i.e. $C = \bigcup_{n\geq 1} \mathrm{Hom}(\mathbf{A}^n, \mathbf{A})$ for some finite algebra $\mathbf{A}$), there exists a category $\mathcal{X}$ of algebraic gadgets such that the ideal $\langle C]$ in the lattice of clones over A is isomorphic to the lattice $\overline{\mathcal{L}}_{\mathbf{X}}$ of clones of dual operations over some object $\mathbf{X} \in \mathcal{X}$. However, $\overline{\mathcal{L}}_{\mathbf{X}}$ is a lattice of coclones if and only if C is isomorphic to the centralizer clone of a boolean algebra, which is obviously a very uncommon occurrence. In all other cases, $\mathbf{X}$ is an object with more structure and the coproducts in $\mathcal{X}$ are not the disjoint union (see [2] for more details). Thus, the Galois theory proposed in [5] is only as general as the usual Galois theory restricted to operations over boolean algebras. Therefore, it is a natural wish to apply the notion of relations, preserving and all the results of the corresponding Galois connection for all dual operations and something that will then be a dual relation.

2. $\overline{\mathrm{Pol}}$-$\overline{\mathrm{Inv}}$

Inspired by this wish, the talk will outline a general Galois Theory for dual operations and what we will call dual relations on an arbitrary (not necessarily finite) object $\mathbf{X}$ in an arbitrary concrete category $\mathcal{X}$ with finite, nonempty coproducts. For the case that $\mathcal{X}$ is the category of sets, it will coincide with the Galois theory in [5].

When we talk about concrete categories, we will use the convention of assuming that the objects are sets with some additional structure and the morphisms are structure preserving maps between them (formally, the objects and morphisms in a concrete category can only be interpreted in this way but do not necessarily have to be of this form).

Denote by $\mathcal{X}^*$ the category we obtain from $\mathcal{X}$ in the natural way if we identify all objects that are isomorphic to each other (the reason for this step will be discussed in the talk).

Definition 8. Let $\mathbf{Y} \in \mathcal{X}^*$ be finite. A *dual relation of type* $\boldsymbol{Y}$ *on* $\boldsymbol{X}$ is a subset of $\mathbf{Y}^{\mathbf{X}}$. Denote the set of all dual relations of type $\mathbf{Y}$ on $\mathbf{X}$ by $\overline{\mathrm{R}}_{\mathbf{X}}^{(\mathbf{Y})}$. For $k \in \mathbb{N}$, let $\overline{\mathrm{R}}_{\mathbf{X}}^{(\mathrm{k})} := \bigcup\limits_{\mathbf{Y}\in\mathcal{X}^*, |\mathbf{Y}|\leq k} \overline{\mathrm{R}}_{\mathbf{X}}^{(\mathbf{Y})}$ and $\overline{\mathrm{R}}_{\mathbf{X}} := \bigcup\limits_{k\geq 1} \overline{\mathrm{R}}_{\mathbf{X}}^{(\mathrm{k})}$.

Definition 9. Let $\sigma \in \overline{\mathrm{R}}_{\mathbf{X}}^{(\mathbf{Y})}$ and let f be an n-ary dual operation over $\mathbf{X}$. We say that σ is *invariant* for f or that f *preserves* σ, written $f \rhd \sigma$, if $[r_1, ..., r_n] \circ f \in \sigma$ whenever $r_1, ..., r_n \in \sigma$.

We define clones of dual relations.

Definition 10. A set $R \subseteq \overline{\mathrm{R}}_{\mathbf{X}}$ is called a *clone of dual relations on* X, written $R \leq \overline{\mathrm{R}}_{\mathbf{X}}$, if $\emptyset \in R$ and R is closed under *general superposition*, i.e. the following holds: Let I be an index set, $\sigma_i \in R^{(\mathbf{Y}_i)}$ $(i \in I)$ and let $\phi : \mathbf{Z} \to \mathbf{Y}$ and $\phi_i : \mathbf{Z} \to \mathbf{Y}_i$ be morphisms where $\mathbf{Y}_i, \mathbf{Z} \in \mathcal{X}^*$ and $\mathbf{Y}_i$ finite. Then the dual relation $\bigwedge^{\phi}_{(\phi_i)_{i \in I}} (\sigma_i)_{i \in I}$ defined by

$$\bigwedge\nolimits^{\phi}_{(\phi_i)_{i \in I}} (\sigma_i)_{i \in I} := \bigwedge\nolimits^{\phi}_{(\phi_i)} (\sigma_i) := \{\phi \circ r \mid \forall\, i \in I : \phi_i \circ r \in \sigma_i,\, r \in \mathbf{Z}^{\mathbf{X}}\}$$

belongs to R.

Definition 11. We define the operators $\overline{\mathrm{Inv}} : \mathfrak{P}(\overline{O}_{\mathbf{X}}) \to \mathfrak{P}(\overline{\mathrm{R}}_{\mathbf{X}})$ and $\overline{\mathrm{Pol}} : \mathfrak{P}(\overline{\mathrm{R}}_{\mathbf{X}}) \to \mathfrak{P}(\overline{O}_{\mathbf{X}})$ as follows: For $F \subseteq \overline{O}_{\mathbf{X}}$ and $R \subseteq \overline{\mathrm{R}}_{\mathbf{X}}$, set

$$\overline{\mathrm{Inv}}\, F := \{\sigma \in \overline{\mathrm{R}}_{\mathbf{X}} \mid \forall\, f \in F : f \rhd \sigma\},$$
$$\overline{\mathrm{Pol}}\, R := \{f \in \overline{O}_{\mathbf{X}} \mid \forall\, \sigma \in R : f \rhd \sigma\}.$$

In this setting, we will be able to find a dual counterpart to almost every definition, lemma, proposition or theorem given in the context of Pol-Inv (see for example [4] and [3]).

Definition 12. Let $F \subseteq \overline{O}_{\mathbf{X}}$, $R \subseteq \overline{\mathrm{R}}_{\mathbf{X}}$ and let $s \geq 1$. We define the following local closure operators:

$$\begin{aligned}\text{s-Loc}\, F :=& \{f \in \overline{O}_{\mathbf{X}}^{(n)} \mid n \geq 1, \forall\, r_1, ..., r_n \in \mathbf{Y}^{\mathbf{X}}, \mathbf{Y} \in \mathcal{X}^*, |\mathbf{Y}| \leq s : \\ & \exists\, f' \in F : [r_1, ..., r_n] \circ f = [r_1, ..., r_n] \circ f'\}, \\ \text{s-LOC}\, R :=& \{\sigma \in \overline{\mathrm{R}}_{\mathbf{X}} \mid \forall B \subseteq \sigma, |B| \leq s : \exists\, \sigma' \in R : B \subseteq \sigma' \subseteq \sigma\}.\end{aligned}$$

Furthermore, let $\mathrm{Loc}\, F := \bigcap_{s \geq 1} \text{s-Loc}\, F$ and $\mathrm{LOC}\, R := \bigcap_{s \geq 1} \text{s-LOC}\, R$.

As the most important result, we can characterize those subsets $F \subseteq \overline{O}_{\mathbf{X}}$ and $R \subseteq \overline{\mathrm{R}}_{\mathbf{X}}$ which can be represented as $\overline{\mathrm{Pol}}\, R'$ and $\overline{\mathrm{Pol}}\, F'$ for some $R' \subseteq \overline{\mathrm{R}}_{\mathbf{X}}$ and $F' \subseteq \overline{O}_{\mathbf{X}}$, respectively.

Theorem 13. *For $F \subseteq \overline{O}_{\mathbf{X}}$, the following are equivalent:*

(1) $F \leq \overline{O}_{\mathbf{X}}$ *and* $\mathrm{Loc}\, F = F$
(2) $F = \overline{\mathrm{Pol}}\, \overline{\mathrm{Inv}}\, F$
(3) $\exists\, R \subseteq \overline{\mathrm{R}}_{\mathbf{X}} : F = \overline{\mathrm{Pol}}\, R$

Theorem 14. *For $R \subseteq \overline{\mathrm{R}}_{\mathbf{X}}$, the following are equivalent:*

(1) $R \leq \overline{\mathrm{R}}_{\mathbf{X}}$ *and* $\mathrm{LOC}\, R = R$
(2) $R = \overline{\mathrm{Inv}}\, \overline{\mathrm{Pol}}\, R$
(3) $\exists\, F \subseteq \overline{O}_{\mathbf{X}} : R = \overline{\mathrm{Inv}}\, F$

If all finite copowers of $\mathbf{X}$ are finite (which happens in most of the usual categories as soon as $\mathbf{X}$ is finite), we have $\operatorname{Loc} F = F$ and $\operatorname{LOC} R = R$ for all $F \subseteq \overline{O}_{\mathbf{X}}$ and $R \subseteq \overline{\mathrm{R}}_{\mathbf{X}}$. Thus, in this case, the Galois closed sets are exactly the clones of dual operations and the clones of dual relations, respectively.

We end the talk with discussing possible applications of this theory.

References

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