Direct decompositions of basic algebras and their idempotent modifications

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1. Introduction

It is well-known that a bounded lattice $\mathcal{L} = (L; \vee, \wedge, 0, 1)$ is directly decomposable into lattices $\mathcal{L}_1, \mathcal{L}_2$ isomorphic to the intervals $[a, 1], [b, 1]$ of $\mathcal{L}$ if b is a complement of a and a, b are standard elements. Since every basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ induces a lattice $\mathcal{L}(A) = (A; \vee, \wedge)$ which is bounded by 0 and $1 = \neg 0$, we can ask if also $\mathcal{A}$ is directly decomposable whenever there exists a complemented and standard element of $\mathcal{L}(A)$. In what follows we show that the condition concerning this element must be enlarged due to the fact that the operations $\oplus$ and $\neg$ cannot be derived by means of the lattice operations of $\mathcal{L}(A)$. However, we set up a natural necessary and sufficient condition for the direct decomposability of $\mathcal{A}$.

By a **basic algebra** (see e.g.[1],[2]) is meant an algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following four axioms

(BA1) $x \oplus 0 = x$;
(BA2) $\neg\neg x = x$;
(BA3) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$;
(BA4) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg 0$.

As usual, we will write 1 instead of $\neg 0$. We say that a basic algebra $\mathcal{A}$ is **non-trivial** if $0 \neq 1$ (i.e. $|A| > 1$).

Having a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$, one can introduce the **induced order** $\leq$ on $\mathcal{A}$ as follows

$$x \leq y \qquad \text{if and only if} \qquad \neg x \oplus y = 1.$$

It is an easy exercise to verify that $\leq$ is really an order on A and $0 \leq x \leq 1$ for each $x \in A$. Moreover, $(A; \leq)$ is a bounded lattice in which

$$x \vee y = \neg(\neg x \oplus y) \oplus y \qquad \text{and} \qquad x \wedge y = \neg(\neg x \vee \neg y).$$

The lattice $\mathcal{L}(A) = (A; \vee, \wedge)$ will be called the **induced lattice** of $\mathcal{A}$. In particular for each $a \in A$ there exists an antitone involution $x \mapsto x^a$ on the interval $[a, 1]$ (called a **section**) where $x^a = \neg x \oplus a$.

It is well-known (see e.g. [1],[3]) that also conversely, if $(A; \vee, \wedge, (^a)_{a \in A}, 0, 1)$ is a bounded lattice with section antitone involutions, we are able to construct a basic algebra using the operations

$$x \oplus y = (x^0 \vee y)^y \qquad \text{and} \qquad \neg x = x^0$$

Lemma 1. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra, $\leq$ the induced order and $a \in A$. Define the polynomial operations $\neg_a$ and $\oplus_a$ on the interval $[a, 1]$ as follows*

$$\neg_a x = \neg x \oplus a \qquad \textit{and} \qquad x \oplus_a y = \neg(\neg x \oplus a) \oplus y.$$

Then $([a, 1]; \oplus_a, \neg_a, a)$ is a basic algebra.

The basic algebra $([a, 1]; \oplus_a, \neg_a, a)$ where the operations $\oplus_a$, $\neg_a$ are defined as in Lemma 1 will be called an **interval basic algebra**.

2. Direct decomposibility of basic algebras

Now, we will set up the conditions under which a basic algebra $\mathcal{A}$ can be directly decomposed. First, we define several concepts.

Definition 2. An element a of a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ is called **strong** if

(a) $x \oplus a = x \vee a$ and $x \oplus \neg a = x \vee \neg a$
for every $x \in A$.

A strong element a of $\mathcal{A}$ is called a **decomposing element** if it moreover satisfies

(b) $(x \oplus y) \oplus a = x \oplus (y \oplus a)$, $(x \oplus y) \oplus \neg a = x \oplus (y \oplus \neg a)$
and $x \oplus a = a \oplus x$, $x \oplus \neg a = \neg a \oplus x$
for all $x, y \in A$.

Let us note that 0 and 1 are decomposing elements for every basic algebra $\mathcal{A}$.

Recall (see [4]) that the element a of a lattice $(L; \vee, \wedge)$ is called **distributive** if for all $x, y \in L$

$$(x \wedge y) \vee a = (x \vee a) \wedge (y \vee a)$$

and the element a of a lattice $(L; \vee, \wedge)$ is called **standard** if for all $x, y \in L$

$$x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y).$$

Further, recall that if $(L; \vee, \wedge)$ is a lattice and $a \in L$ then the following two conditions are equivalent:

(α) a is standard
(β) a is distributive and, for $x, y \in L$,

$$a \wedge x = a \wedge y \quad \text{and} \quad a \vee x = a \vee y \qquad \text{imply that} \qquad x = y$$

(for more details see [4]).

Lemma 3. *Let a be a strong element of a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$. Then*

(i) *a is boolean (i.e. $a \vee \neg a = 1$, $a \wedge \neg a = 0$);*
(ii) *a and $\neg a$ are distributive elements.*

Lemma 4. *Let a be a strong element of a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ and $\neg a$ be a standard element of the induced lattice $\mathcal{L}(A) = (A; \vee, \wedge)$. Then the mapping $\varphi_a(x) = (x \vee a, x \vee \neg a)$ is a lattice isomorphism of $\mathcal{L}(A)$ onto the direct product of lattices $([a, 1]; \vee, \wedge) \times ([\neg a, 1]; \vee, \wedge)$.*

Theorem 5. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra. Then $\mathcal{A}$ is isomorphic to a direct product of non-trivial basic algebras $\mathcal{B}_1, \mathcal{B}_2$ if and only if there exists a decomposing element $a \in A$, $0 \neq a \neq 1$ such that $\neg a$ is standard in the induced lattice $\mathcal{L}(A) = (A; \vee, \wedge)$. If it is the case then $\mathcal{A}$ is isomorphic to the direct product of interval basic algebras $([a, 1]; \oplus_a, \neg_a, a)$ and $([\neg a, 1]; \oplus_{\neg a}, \neg_{\neg a}, \neg a)$.*

3. Idempotent modification of basic algebras

The concept of idempotent modification of an algebra was introduced by J. Ježek [6] as follows.

Definition 6. An **idempotent modification** of an algebra $\mathcal{A} = (A; F)$ is an algebra $\mathcal{A}_I = (A; F_I)$ with the same underlying set A, where $|F| = |F_I|$ and for every $f \in F$ the corresponding operation $f_I \in F_I$ is defined as follows

(i) if f is at most unary then $f_I = f$;
(ii) if f is n-ary with $n > 1$ and $a_1, \dots, a_n \in A$ then

$$f_I(a_1, \dots, a_n) = \begin{cases} a_1 & \text{if} \quad a_1 = a_2 = \dots = a_n \\ f(a_1, \dots, a_n) & \text{otherwise.} \end{cases}$$

The main result of [6] is that for any group G its idempotent modification G_I is subdirectly irreducible.

In what follows we will treat direct decomposability of an idempotent modification of a basic algebra.

For this we slightly modify our definition of basic algebra. As mentioned above, every basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ has induced lattice $\mathcal{L}(A) = (A; \vee, \wedge)$ where $\vee$ and $\wedge$ are term operations of $\mathcal{A}$. Hence, inserting $\vee$ and $\wedge$ into the type of $\mathcal{A}$, we obtain an algebra with the same clone of term operations and hence term equivalent to $\mathcal{A}$. From now on, by a basic algebra we will understand an algebra $\mathcal{A} = (A; \oplus, \neg, 0, \vee, \wedge)$ where the term operations $\vee$ and $\wedge$ are defined by $x \vee y = \neg(\neg x \oplus y) \oplus y$, $x \wedge y = \neg(\neg x \vee \neg y)$.

The reason of this insertion is that when an idempotent modification of $(A; \oplus, \neg, 0)$ is considered, the resulting algebra does not have the lattice structure. However, if $\mathcal{A} = (A; \oplus, \neg, 0, \vee, \wedge)$ is treated then the lattice structure for $\mathcal{A}_I$ is preserved because both $\vee$ and $\wedge$ are idempotent operations on A.

Theorem 7. *Let $\mathcal{A} = (A; \oplus, \neg, 0, \vee, \wedge)$ be a basic algebra whose at least one element is not boolean. Then its idempotent modification $\mathcal{A}_I = (A; \oplus_I, \neg, 0, \vee, \wedge)$ is not directly decomposable.*

Call a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ **distributive** if the induced lattice $\mathcal{L}(A) = (A; \vee, \wedge)$ is distributive. For example, if $\mathcal{A}$ is commutative then $\mathcal{A}$ is distributive (but not vice versa) see e.g. [1]. For distributive basic algebras, we can modify our result as follows

Corollary 8. *Let $\mathcal{A} = (A; \oplus, \neg, 0, \vee, \wedge)$ be a distributive basic algebra with $|A| > 2$. Then its idempotent modification is directly indecomposable if and only if $\mathcal{A}$ contains an element which is not boolean.*

References

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