Note on strict residuated lattices with an involutive negation

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1. Introduction

In [1], BL-algebras with a new involutive negation $\neg$ are introduced and studied, where the new negation is different from the original one $x' = x \to 0$ defined by its t-norm $\odot$. The new involutive negation $\neg$ characterized by

($\neg$1) $\neg\neg x = x$
($\neg$2) $x' \leq \neg x$
($\neg$3) $\Delta(x \to y) = \Delta(\neg y \to \neg x)$, where $\Delta x = \neg x \to 0$
($\neg$4) $\Delta x \vee (\Delta x)' = 1$
($\neg$5) $\Delta(x \vee y) \leq \Delta x \vee \Delta y$
($\neg$6) $\Delta x \odot \Delta(x \to y) \leq \Delta y$

is introduced on strict BL-algebras (simply SBL-algebras), that is, BL-algebras with a strictness axiom $(S) : (x \odot y)' = x' \vee y'$. It is also considered independence of these axioms in [1] and is proved that, for any SBL-algebra $(L, \wedge, \vee, \odot, \to, 0, 1)$, it is an SBL$_\neg$-algebra if and only if it satisfies only ($\neg$1), ($\neg$3) and ($\neg$6). The following question was left open in it:

> Are the axioms ($\neg$1), ($\neg$3) and ($\neg$6) over SBL-algebras independent?

This was recently solved [5] that, for any SBL-algebra, it is an SBL$_\neg$-algebra if and only if it satisfies ($\neg$1) and ($\neg$3).

Since MV-algebras are axiomatic extensions of BL-algebras, the result above also holds for MV-algebras, that is, for any strict MV-algebra, it is an SMV$_\neg$-algebra if and only if it satisfies ($\neg$1) and ($\neg$3).

It is a natural question that "How about the case of MTL-algebras?" or generally "For any strict bounded commutative residuated lattice (simply call an SRL-algebra), is it true that it is an SRL$_\neg$-algebra if and only if it satisfies ($\neg$1) and ($\neg$3) ?"

We answer the question "yes" that

> For any SRL-algebra, it is an SRL$_\neg$-algebra if and only if it satisfies ($\neg$1) and ($\neg$3).

2. Strict bounded commutative residuated lattice (SRL)

An algebraic structure $\mathcal{L} = (L, \wedge, \vee, \odot, \to, 0, 1)$ is called an integral bounded commutative residuated lattice (simply called *residuated lattice* and denoted by RL here) if

(1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice;
(2) $(L, \odot, 1)$ is a commutative monoid;
(3) For all $x, y, z \in L$,

$$x \odot y \leq z \text{ if and only if } x \leq y \to z$$

Some well-known algebras, MTL-algebras, BL-algebras, MV-algebras and so on, are axiomatic extensions of RL as follows:

$$\mathrm{MTL} = \mathrm{RL} + \{(x \to y) \vee (y \to x) = 1\}$$
$$\mathrm{BL} = \mathrm{MTL} + \{x \wedge y = x \odot (x \to y)\}$$
$$\mathrm{MV} = \mathrm{BL} + \{x'' = x\}$$

An integral bounded commutative residuated lattice L is called *strict* (simply called an SRL-algebra) if it satisfies an axiom called strictness

(S): $(x \odot y)' = x' \vee y'$, where $x' = x \to 0$.

For the sake of simplicity, we denote SMTL, SBL and SMV, for algebras MTL, BL and MV with strict axiom. Thus, for example, SBL is a strict BL-algebra, that is, BL-algebra with meeting the axiom $(x \odot y)' = x' \vee y'$.

Proposition 1. *Let L be an SRL-algebra. For all $x, y, z \in L$,*

(1) $x \odot x' = 0$, *hence* $x' \vee x'' = 1$
(2) $x \leq x''$, $x' = x'''$
(3) $x \to x' = x'$
(4) $x' \to x = x''$
(5) $(x \to y)'' = y' \to x'$

In [3], for linearly ordered BL-algebras, it is proved that the strictness axiom $(x \odot y)' = x' \vee y'$ is equivalent to the condition that the negation is a Gödel one, that is, $x' = 0$ if $x \neq 0$ and $x' = 1$ if $x = 0$. We can show a simila r result in the case of linearly ordered RL-algebras (i.e., RL-chains).

Proposition 2. *On any linearly ordered RL-algebra L, the following conditions are equivalent:*

(1) (S) : $(x \odot y)' = x' \vee y'$
(2) *If* $x \odot y = 0$ *then* $x = 0$ *or* $y = 0$
(3) $x' = x \to 0$ *is the Gödel negation.*

We note that, for any SRL-algebra, the negation has a property that $x' = 1$ if and only if $x = 0$, which is also proved in [4] in the case of SBL-algebras.

According to [1], we introduce a new involutive negation $\neg$.

($\neg$1) $\neg\neg x = x$
($\neg$2) $x' \leq \neg x$
($\neg$3) $\Delta(x \to y) = \Delta(\neg y \to \neg x)$, where $\Delta x = \neg x \to 0$
($\neg$4) $\Delta x \vee (\Delta x)' = 1$
($\neg$5) $\Delta(x \vee y) \leq \Delta x \vee \Delta y$
($\neg$6) $\Delta x \odot \Delta(x \to y) \leq \Delta y$

For an SRL-algebra, we call it an SRL$_\neg$-algebra if it contains a new symbol $\neg$ as a language and satisfies ($\neg$1) - ($\neg$6) above. We similarly define SMTL$_\neg$-, SBL$_\neg$- and SMV$_\neg$-algebras. Thus, for example, an SMTL_$\neg$-algebra is a strict MTL-algebra satisfying ($\neg$1) - ($\neg$6).

Proposition 3. *Let L be an SRL-algebra satisfying only (¬1) and (¬3). Then we have*

(1) $\neg 0 = 1,\ \neg 1 = 0$
(2) $\Delta x = 1 \Longleftrightarrow x = 1$
(3) $x \leq y \Longrightarrow \neg y \leq \neg x$
(4) $x \leq y \Longrightarrow \Delta x \leq \Delta y$
(5) $\Delta x \leq x''$
(6) $\Delta x' = x'$
(7) $\Delta\Delta x = \Delta x$

In [4], [5], for every SBL-algebra, it is an SBL$_\neg$-algebra if and only if it satisfies (¬1) and (¬3). We can show that the result also holds in the case of RL-algebras. This result generalizes the one obtained in [4], [5].

Theorem 4. *For every SRL-algebra, if it is an SRL$_\neg$-algebra if and only if it satisfies the axioms (¬1) and (¬3).*

Moreover we see that (¬6) is derived from (¬2).

Proposition 5. *For any SRL$_\neg$-algebra which is defined by (¬1) and (¬3), we have*

(a) (¬2) $\Longleftrightarrow$ $\Delta x \leq x$ *and*
(b) (¬6) $\Longleftrightarrow$ $\Delta x \leq x''$.

Thus (¬6) can be obtained from (¬2)

Corollary 6. *For every SMTL-algebra, it is an SMTL$_\neg$-algebra if and only if it satisfies (¬1) and (¬3).*

Theorem 7. *For every SRL$_\neg$-algebra L, the following conditions are equivalent:*

(1) Δ *is an identity map;*
(2) Δ *is a homomorphism;*
(3) $\Delta(x \to y) = \Delta \to \Delta y$;
(4) $(\Delta x)' = \Delta x'$;
(5) $\neg x = x'$;
(6) $x \odot \neg x = 0$;
(7) L *is a Boolean algebra,* $\odot = \wedge$ *and* $\neg ='$

References

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