Anti-inverse Subsemigroups of the Full Transformation Semigroup

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The notion of anti-inverse elements in a semigroup was introduced by J.C. Sharp in 1977 ([6]). In [2], Bogdanović, Milić, and Pavlović studied the structure and considered some properties of those semigroups. In 1982, the anti-inverse semigroups were studied by Blagojevich ([1]). An element a of a semigroup S is called anti-inverse if there exists an element $b \in S$ such that $aba = b$ and $bab = a$ (in this case a and b are called mutual anti-inverse). The semigroup S is called anti-inverse if each element of S is anti-inverse.

For particular classes of semigroups, the anti-regular elements are studied (see [2], [6]). Clearly, each band is anti-inverse because of the idempotent low. Moreover, each element of a band has a unique anti-inverse element. An abelian group is anti-inverse if and only if each element is inverse to itself. But not each group is anti-inverse. An abelian semigroup S is anti-inverse if and only if $x = x^3$ for each element $x \in S$.

We want to initiate the study of anti-inverse transformation semigroups. Here, we characterize the anti-inverse elements of the full transformation semigroup T_n. A $\mathcal{J}$-class in T_n consists of all transformations with the same rank. Moreover, we determine all anti-inverse semigroups in the $\mathcal{J}$-classes. Let note that the semigroup of all order-preserving (order-preserving or order-reversing) transformations as well as the semigroup of all orientation-preserving (orientation-preserving or orientation-reversing) transformations are of particular interest (see for example [3], [4], [5]). In order to illustrate our result, we will describe the anti-inverse semigroups within the $\mathcal{J}$-classes of these transformation semigroups. The last part is devoted semigroups with elements of order 2. We give a description of its maximal anti-inverse subsemigroups containing particular transformations with rank ≤ 3.

We will try to keep the standard notation. Let $X_n = \{1 < \cdots < n\}$ be a finite chain with n - elements. The *full transformation semigroup* T_n is the set of all mappings, written on the right, of X_n into itself with the composition of mappings as multiplication. For every transformation $\alpha \in T_n$, by $\ker \alpha$ and $im\alpha$ we denote the kernel and the image of α, respectively.

Each $\mathcal{J}$-class of T_n has the form

$$J_k := \{\alpha \in T_n : |im\ \alpha| = k\} \text{ for } 1 \leq k \leq n.$$

For $1 \leq k \leq n$ let us denote by Λ_k the collection of all subsets of X_n of cardinality k. Let $A \in \Lambda_k$ and let π be an equivalence relation on X_n with

$|X_n/\pi| = k$. We say that A *is a transversal of* π if $|A \cap \bar{x}| = 1$ for every equivalence class $\bar{x}$ of π. Also we may write $A \# \pi$.

An $\mathcal{H}$-class of an element $\alpha \in T_n$ consists of all transformations with the same image and the same kernel as α and will be denoted by H_α. Let U be a subset of T_n. We denote by $E(U)$ the set of all idempotents in the set U.

Let $\alpha \in T_n$ be an anti-inverse element. Then there exists an element $\beta \in T_n$ such that $\alpha\beta\alpha = \beta$ and $\beta\alpha\beta = \alpha$. These equations imply $im\alpha = im\beta$, $\ker\alpha = \ker\beta$, and $im\alpha \# \ker\beta$. Therefore, we have that $\alpha, \beta \in H_\alpha$ and H_α contains an idempotent.

Definition 1. For $\varepsilon \in E(T_n)$ we put

$$H_\varepsilon^* := \{\alpha \in H_\varepsilon : \text{ there is a } \beta \in H_\varepsilon \text{ with } \alpha^4 = \beta^4 = \alpha^2\beta^2 = \alpha\beta\alpha\beta^3 = \varepsilon\}.$$

Proposition 2. *Let S be a subsemigroup of T_n. Then S is an anti-inverse semigroup if and only if $S \subseteq \bigcup\{H_\varepsilon^* : \varepsilon \in E(T_n)\}$.*

Proposition 2 shows that the anti-inverse subsemigroups of T_n are Clifford semigroups (completely regular semigroups).

Let $1 \leq k \leq n \in N$ and $S \subseteq J_k$. Then S is a semigroup in J_k iff there are congruence relations $\pi_1, \ldots, \pi_r$ on X_n with $|X_n/\pi_i| = k$ $(i = 1, \ldots, r)$ and sets $A_1, \ldots, A_s \in \Lambda_k$ such that $A_j \# \pi_i$ for $1 \leq i \leq r$ and $1 \leq j \leq s$, and $S = \{\alpha \in J_k : \ker\alpha \in \{\pi_1, \ldots, \pi_r\} \text{ and } im\alpha \in \{A_1, \ldots, A_s\}\}$.

Theorem 3. *Let $1 \leq k \leq n \in N$ and $S \subseteq J_k \subseteq T_n$. Then S is an anti-inverse semigroup iff there is a semigroup $\widetilde{S}$ in $E(T_n) \cap J_k$ and a semigroup K in H_ε^* for some $\varepsilon \in \widetilde{S}$ such that $S = \bigcup\{\{\widetilde{\varepsilon}\alpha\widetilde{\varepsilon} : \alpha \in K\} : \widetilde{\varepsilon} \in \widetilde{S}\}$.*

A transformation $\alpha \in T_n$ is called *order-preserving* (respectively, *order-reversing*) if for all $x, y \in X_n$, $x \leq y$ implies $x\alpha \leq y\alpha$ (respectively, $x\alpha \geq y\alpha$).

The set $O_n := \{\alpha \in T_n : \alpha \text{ is order-preserving}\}$ forms a subsemigroup of T_n, which is called *the semigroup of all order-preserving transformations* (see [5]).

The set $M_n := \{\alpha \in T_n : \alpha \text{ is order-preserving or order-reversing}\}$ forms a subsemigroup of T_n, which is called the *semigroup of all order-preserving or order-reversing transformations* (see [4]).

It is clear that the semigroups O_n and M_n are regular subsemigroups of T_n. Moreover, $O_n \subseteq M_n \subseteq T_n$. Let $\alpha \in M_n$. Then the $(\ker\alpha)$ - classes are convex subsets C of X_n, in the sense that $x, y \in C$ and $x \leq z \leq y$ together imply that $z \in C$.

Let $1 \leq k \leq n \in N$ and $\varepsilon \in E(T_n) \cap J_k$ with

$\varepsilon = \begin{pmatrix} \overline{x}_1 & \ldots & \overline{x}_k \\ a_1 & \ldots & a_k \end{pmatrix}$ and $a_1 < \cdots < a_k$. Then we put

$\varepsilon^a := \begin{pmatrix} \overline{x}_1 & \ldots & \overline{x}_k \\ a_k & \ldots & a_1 \end{pmatrix}$, i.e. $\overline{x}_i\varepsilon^a = a_{k-i+1}$ for $1 \leq i \leq k$. Clearly, $(\varepsilon^a)^2 = \varepsilon$.

Proposition 4. *Let $1 \leq k \leq n \in N$ and $S \subseteq J_k \subseteq M_n$. Then S is an anti-inverse semigroup iff there is a semigroup $\widetilde{S}$ in $E(T_n) \cap J_k$ such that $S = \widetilde{S}$ or $S = \bigcup\{\{\varepsilon, \varepsilon^a\} : \varepsilon \in \widetilde{S}\}$.*

Corollary 5. *Let $1 \leq k \leq n \in N$ and $S \subseteq J_k \subseteq O_n$. Then S is an anti-inverse semigroup iff S is a semigroup in $E(T_n) \cap J_k$.*

Now, let $a = (a_1, a_2, \ldots, a_t)$ be a sequence of $t(t \geq 1)$ elements from the chain X_n. We say that a is *cyclic* (respectively, *anti-cyclic*) if there exists no more than one index $i \in \{1, \ldots, t\}$ such that $a_i > a_{i+1}$ (respectively, $a_i < a_{i+1}$), where a_{t+1} denotes a_1. A transformation $\alpha \in T_n$ is called *orientation-preserving* (respectively, *orientation-reversing*) if the sequence of its images is cyclic (respectively, anti-cyclic).

The set $OP_n := \{\alpha \in T_n : \alpha \text{ is orientation-preserving } \}$ forms a subsemigroup of T_n, which is called *the semigroup of all orientation-preserving transformations* (see [3]).

The set $OPR_n := \{\alpha \in T_n : \alpha \text{ is orientation-preserving or orientation-reversing } \}$ forms a subsemigroup of T_n, which is called the *semigroup of all orientation-preserving or orientation-reversing transformations* (see [4]). Moreover, we put $OR_n := OPR_n \setminus OP_n$.

Let $\alpha \in OPR_n$. Then for all $\overline{x} \in X_n/\ker\alpha$, the set $\overline{x}$ or $X_n \setminus \overline{x}$ is convex. Let $1 \leq k \leq n \in N$ and $\varepsilon \in E(T_n) \cap J_k$ with
$\varepsilon = \begin{pmatrix} \overline{x}_1 & \cdots & \overline{x}_k \\ a_1 & \cdots & a_k \end{pmatrix}$ where $a_1 < \cdots < a_k$.
If $k \in 2N$ then we put ε' as the orientation-preserving transformation in H_ε with

$$\overline{x}_p\varepsilon' = \begin{cases} a_{p+\frac{k}{2}} & \text{if} \quad 1 \leq p \leq \frac{k}{2} \\ a_{p-\frac{k}{2}} & \text{if} \quad \frac{k}{2} < p \leq k. \end{cases}$$

Clearly, $(\varepsilon')^2 = \varepsilon$. If k is odd then $H^*_\varepsilon = \{\varepsilon\}$. If $k \in 2N$ then $H^*_\varepsilon = \{\varepsilon, \varepsilon'\}$.

Proposition 6. *Let $1 \leq k \leq n \in N$ and $S \subseteq J_k \subseteq OPR_n$. Then S is an anti-inverse semigroup iff there is a semigroup $\widetilde{S} \subseteq E(T_n) \cap J_k$ such that*
(i) $S = \widetilde{S}$ or
(ii) $S = \bigcup\{\{\varepsilon, \varepsilon\alpha\varepsilon\} : \varepsilon \in \widetilde{S}\}$ for some $\alpha \in OR_n \cap H^*_{\varepsilon_\alpha}$ with $\varepsilon_\alpha \in \widetilde{S}$ or
(iii) $S = \bigcup\{\{\varepsilon, \varepsilon'\} : \varepsilon \in \widetilde{S}\}$ and k is even or
(iv) $S = \bigcup\{\{\varepsilon, \varepsilon', \varepsilon\alpha\varepsilon, \varepsilon\alpha\varepsilon'_\alpha\varepsilon\} : \varepsilon \in \widetilde{S}\}$ for some $\alpha \in OR_n \cap H^*_{\varepsilon_\alpha}$ where $\varepsilon_\alpha \in \widetilde{S}$ and k is even.

Corollary 7. *Let $1 \leq k \leq n \in N$ and $S \subseteq J_k \subseteq OP_n$. Then S is an anti-inverse semigroup iff there is a semigroup $\widetilde{S} \subseteq E(T_n) \cap J_k$ such that*
(i) $S = \widetilde{S}$ or
(ii) $S = \bigcup\{\{\varepsilon, \varepsilon'\} : \varepsilon \in \widetilde{S}\}$ and k is even.

References

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