A category of L-Chu correspondences and Galois functor

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An L-fuzzy Formal Concept Analysis (FCA) provides new space for research of categorical properties of FCA. This presentation will show some differences of categorical properties of classical FCA and L-fuzzy FCA.

Main notion of FCA is a formal context and formal concept. L-fuzzy extension of FCA was provided by Bělohlávek [1].

Definition 1. *Let L be a complete residuated lattice, an **L-fuzzy formal context** is a triple $\langle B, A, r\rangle$ consisting of a set of objects B, a set of attributes A and an L-fuzzy binary relation r, i.e. a mapping from $r\colon B \times A$ to L, which can be alternatively understood as an L-fuzzy subset of $B \times A$*

Definition 2. *A **complete residuated lattice** is an algebra $\langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$ where*

(1) $\langle L, \wedge, \vee, 0, 1\rangle$ is a lattice with the least element 0 and the greatest element 1,
(2) $\langle L, \otimes, 1\rangle$ is a commutative monoid,
(3) $\otimes$ and $\to$ are adjoint, i.e. $a \otimes b \leq c$ if and only if $a \leq b \to c$, for all $a, b, c \in L$ ($\leq$ is the lattice ordering generated from $\wedge$ and $\vee$).

Definition 3. *Consider an L-fuzzy context $\langle B, A, r\rangle$. A pair of mappings $\uparrow\colon L^B \to L^A$ and $\downarrow\colon L^A \to L^B$ is defined as follows:*

$$\begin{aligned}\uparrow(f)(a) &= \bigwedge_{o\in B} (f(o) \to r(o,a))\\ \downarrow(g)(o) &= \bigwedge_{a\in A} \big(g(a) \to r(o,a)\big).\end{aligned}$$

for every $f \in L^B$ and $g \in L^A$.

Bělohlávek also proved that the pair of mappings $\langle\uparrow\downarrow\rangle$ forms a Galois connection between the complete lattices of all L-subsets of the set of objects and attributes.

Definition 4. An **L-fuzzy concept** is a pair $\langle f, g\rangle$ such that $\uparrow f = g, \downarrow g = f$. The first component f is said to be the **extent** of the concept, whereas the second component g is the **intent** of the conce pt.

The set of all L-fuzzy concepts associated to a fuzzy context $\langle B, A, r\rangle$ will be denoted as $CL_L(B, A, r)$.

An ordering between L-fuzzy concepts is defined as follows: $\langle f_1, g_1\rangle \leq \langle f_2, g_2\rangle$ if and only if $f_1 \subseteq f_2$ if and only if $g_1 \supseteq g_2$.

L-fuzzy extension of main theorem on concept lattices.

Theorem 5. *$CL_L(B, A, r) = \{\langle f, g\rangle | \uparrow (f) = g, \downarrow (g) = f\}$ is under $\leq$ ($\langle f_1, g_1\rangle \leq \langle f_2, g_2\rangle$ iff $f_1 \subseteq f_2$ iff $g_1 \supseteq g_2$) a complete lattice where*

$$\bigwedge_{j\in J}\langle f_j, g_j\rangle = \Big\langle \bigwedge_{j\in J} f_j, \uparrow (\bigwedge_{j\in J} f_j)\Big\rangle,$$

$$\bigvee_{j\in J}\langle f_j, g_j\rangle = \Big\langle \downarrow (\bigwedge_{j\in J} g_j), \bigwedge_{j\in J} g_j\Big\rangle.$$

Hence $\uparrow$ and $\downarrow$ are dual isomorphisms between the lattices of $\uparrow\downarrow$-closed L-fuzzy sets in A and $\downarrow\uparrow$-closed L-fuzzy sets in B.

Hideo Mori in [4] defined so called *Chu correspondences* as a pair of mappings between two formal contexts. In [3] is showed an L-fuzzy extension of Chu correspondences between two L-fuzzy contexts.

Definition 6. Consider two L-fuzzy contexts $C_i = \langle B_i, A_i, r_i\rangle, (i = 1, 2)$, then the pair $\varphi = (\varphi_l, \varphi_r)$ is called a **correspondence** from C_1 to C_2 if φ_l and φ_r are L-multifunctions, respectively, from B_1 to B_2 and from A_2 to A_1 (that is, $\varphi_l : B_1 \to L^{B_2}$ and $\varphi_r : A_2 \to L^{A_1}$).

The L-correspondence φ is said to be a **weak L-Chu correspondence** if the equality $\hat{r}_1(\chi_{o_1}, \varphi_r(a_2)) = \hat{r}_2(\varphi_l(o_1), \chi_{a_2})$ holds for all $o_1 \in B_1$ and $a_2 \in A_2$. By unfolding the definition of $hatr_i$ this means that

$$\bigwedge_{a_1\in A_1} (\varphi_r(a_2)(a_1) \to r_1(o_1, a_1)) = \bigwedge_{o_2\in B_2} (\varphi_l(o_1)(o_2) \to r_2(o_2, a_2)) \tag{1}$$

A weak Chu correspondence φ is an **L-Chu correspondence** if $\varphi_l(o_1)$ is closed in C_2 and $\varphi_r(a_2)$ is closed in C_1 for all $o_1 \in B_1$ and $a_2 \in A_2$.

Mori also proved that all Chu correspondences between two formal contexts with a multifunctional ordering form a complete lattice. Same fact about L-Chu correspondences is proved in [3]. Mori showed a category of Chu correspondences between for mal contexts so called ChuCors. Mori also defined a Galois functor from ChuCors to Slat a category of *supremum preserving mappings between complete lattices* and showed that the functor is full, faithfull and equivalence functor betwee n ChuCors and Slat.

L-Chu correspondeces between L-contexts form a category so called L-ChuCors. L-Galois functor is an L-fuzzy extended Galois functor applicable on L-ChuCors, such that to every L-context assigns its L-concept lattice and to every L-Chu correspondence assigns a supremum preserving mapping. My future work is to show fullness and faithfulness of L-Galois functor similarly as Mori's classical one. A difference between L-fuzzy case and classical case of this categorical view on FCA is in equivalence between ChuCors and Slat and non-equivalence between L-ChuCors and Slat.

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