Pre-ideals of basic algebras

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A *basic algebra* is an algebra $(A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ that satisfies the identities

$$x \oplus 0 = x,$$
$$\neg\neg x = x,$$
$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x,$$
$$\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg 0.$$

Basic algebras were introduced in [4] as a counterpart of bounded lattices with sectional antitone involutions and can be regarded as a common generalization of MV-algebras and orthomodular lattices. More precisely, for every basic algebra A, the underlying order defined by $x \leq y$ iff $\neg x \oplus y = \neg 0$ makes A into a bounded lattice where, for each $a \in A$, the map $x \mapsto \neg x \oplus a$ is an antitone involution on the principal filter $[a)$. On the other hand, if we are given a bounded lattice with sectional antitone involutions $x \mapsto x^a$ $(a \in A)$, then the rule $x \oplus y := (x^0 \vee y)^y$ and $\neg x := x^0$ defines a basic algebra. Concerning the interconnection between basic algebras, MV-algebras and orthomodular lattices, MV-algebras are precisely the associative basic algebras, and orthomodular lattices may be characterized as basic algebras satisfying the identity $x \oplus (x \wedge y) = x$. Also lattice effect algebras are a special case of basic algebras (see [4]).

We study what we call *pre-ideals* of basic algebras, that is, non-empty subsets that are closed under $\oplus$ and downwards closed. Since the variety of basic algebras is ideal determined, the term 'ideal' is reserved for the congruence kernels. It is not hard to show that the ideal lattice forms a sublattice of the lattice of pre-ideals.

We restrict ourselves to basic algebras satisfying the identity

$$x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z) \tag{1}$$

or, more generally, the identity

$$\neg x \oplus (x \wedge y) = \neg x \oplus y. \tag{2}$$

These algebras are much closer to MV-algebras; for instance, the induced lattice is distributive, and we prove that finite basic algebras satisfying (2) is automatically an MV-algebra.

Theorem 1. *Every basic algebra satisfying (2) has the following Riesz decomposition property: If $a \leq \tau(b_1, \ldots, b_n)$, where τ is an additive term, then there exist $c_i \leq b_i$ such that $a = \tau(c_1, \ldots, c_n)$.*

Let $(A, \oplus, \neg, 0)$ be a basic algebra that satisfies (2). For every $a \in A$ and integer $n \geq 0$ we define $n \otimes a$ inductively: $0 \otimes a := 0$ and $n \otimes a := a \oplus ((n-1) \otimes a)$ for $n \geq 1$.

If A is *finite*, then for every atom $a \in A$, the set $N(a) = \{n \otimes a \mid n \geq 0\}$ is a finite chain $0 < a < \cdots < \hat{a}$. The Riesz decomposition property entails $N(a) = [0, \hat{a}]$. Since intervals in basic algebras are basic algebras, it fol lows that $N(a)$ is a finite MV-chain.

Theorem 2. *Let A be a finite basic algebra satisfying (2). Then A is isomorphic to the direct product $\prod_{a \in M} N(a)$ where M denotes the set of all atoms of A. Consequently, A is an MV-algebra.*

Remark 3. In a finite basic algebra satisfying (2), every element x can uniquely be expressed as $\bigvee_{a \in M} x_a$ where x_a is a suitable multiple of a (in fact, $x_a = x \wedge \hat{a}$).

Further, given a basic algebra A and its pre-ideal I, we define a binary relation θ_I on A as follows: $(x, y) \in \theta_I$ iff $x = a_1 \oplus (\cdots \oplus (a_m \oplus (y \wedge c)) \dots)$ and $y = b_1 \oplus (\cdots \oplus (b_n \oplus (x \wedge d)) \dots)$ for some $a_i, b_j \in I$ and $c, d \in A$.

Theorem 4. *Let A be a basic algebra satisfying (2) and I a pre-ideal of A. Then θ_I is an equivalence compatible with the meet-operation; the underlying order of A/θ_I is given by: $[x]_{\theta_I} \leq [y]_{\theta_I}$ iff $x = a_1 \oplus (\cdots \oplus (a_m \oplus (y \wedge c)) \dots)$ for some $a_i \in I$ and $c \in A$. Moreover, if I is the kernel of a congruence ϕ, then $\theta_I = \phi$.*

We define a pre-ideal P of a basic algebra A to be *prime* if for all $x, y \in A$, whenever $x \wedge y \in P$, then $x \in P$ or $y \in P$.

Theorem 5. *Let A be a basic algebra that satisfies (2). Then for every pre-ideal P, the following are equivalent:*

(1) P is prime;
(2) for all $x, y \in A$, if $\neg(\neg x \oplus y) \in P$ or $\neg(\neg y \oplus x) \in P$;
(3) $(A/\theta_I, \leq)$ is a chain.

We can prove more for basic algebras satisfying (1). First, for every $\emptyset \neq X \subseteq A$, the pre-ideal $I(X)$ generated by X consists of those $a \in A$ which are less than or equal to some finite sum of elements of X. For all $x, y \in A$ we have $I(x) \cap I(y) = I(x \wedge y)$.

Theorem 6. *The lattice of pre-ideals of a basic algebra satisfying the identity (1) is an algebraic distributive lattice.*

Theorem 7. *Let A be a basic algebra satisfying (1). Then for every pre-ideal P, the following are equivalent:*

(1) P is prime;
(2) P is meet-irreducible in the lattice of pre-ideals;
(3) the set of all pre-ideals containing P is a chain under inclusion.

References

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