Hyperidentities and Hypersubstitutions of Many-Sorted Algebras

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The purpose of this paper is to apply the theory of conjugate pairs of additive closure operators to many-sorted algebras.

If A and B are sets and if $R \subseteq A \times B$ is a binary relation between A and B, then the operators ι and μ defined between the power sets $\mathcal{P}(A)$ and $\mathcal{P}(B)$ of A and B; respectively, by $\iota(S) := \{t \mid t \in B \text{ and } \forall s \in S((s,t) \in R)\}$ and $\mu(T) := \{s \mid s \in A \text{ and } \forall t \in T((s,t) \in R)\}$ for $S \subseteq A, T \subseteq B$ forms a Galois connection between A and B. A pair $\gamma = (\gamma_1, \gamma_2)$ of closure operators $\gamma_1 : \mathcal{P}(A) \to \mathcal{P}(A)$, $\gamma_2 : \mathcal{P}(B) \to \mathcal{P}(B)$ is said to be conjugate with respect to R if for all $s \in A, t \in B$ we have $(\gamma_1(s), t) \in R \Leftrightarrow (s, \gamma_2(t)) \in R$ and additive if $\gamma_1(S) = \bigcup\limits_{s \in S} \gamma_1(\{s\})$ (and $\gamma_2(T) = \bigcup\limits_{t \in T} \gamma_2(\{t\})$). The relation $R_\gamma := \{(s, \gamma_2(\{t\})) \mid s \in A, t \in B\}$ $(= \{(\gamma_1(\{s\}), t) \mid s \in A, t \in B\})$ defines a second Galois connection $(\iota_\gamma, \mu_\gamma)$ between A and B. The theory of conjugate pairs of additive closure operators describes the relationships between the following 6 complete lattices

$$
\begin{aligned}
&\mathcal{H}_{\iota\mu} := \{t \in B \mid (\iota\mu)(t) = t\},\\
&\mathcal{H}_{\mu\iota} := \{s \in A \mid (\mu\iota)(s) = s\},\\
&\mathcal{H}_{\iota_\gamma\mu_\gamma} := \{t \in B \mid (\iota_\gamma\mu_\gamma)(t) = t\},\\
&\mathcal{H}_{\mu_\gamma\iota_\gamma} := \{s \in A \mid (\mu_\gamma\iota_\gamma)(s) = s\},\\
&\mathcal{S}_{\gamma_1} := \{s \in A \mid \gamma_1(s) = s\},\\
&\mathcal{S}_{\gamma_2} := \{t \in A \mid \gamma_2(t) = t\}.
\end{aligned}
$$

This theory was applied to the equational theory of Universal Algebra and led to the study of M-hyperidentities and M-solid varieties of partial algebras, to M-hyperquasiidentities and M-solid quasivarieties, to M-hyperpseudoidentities and M-solid pseudovarieties and to hyperformulas and solid model classes.

The concept of many-sorted algebras or Σ-algebras was first introduced by P. J. Higgins as an ordered pair $\mathcal{A} := (A; ((f_\gamma)_k)^{\mathcal{A}})_{k \in K_\gamma, \gamma \in \Sigma})$, where A is an I-sorted non-empty set and a set of finitary operations $((f_\gamma)_k)^{\mathcal{A}} : A_{k_1} \times \cdots \times A_{k_n} \to A_i$ defined on A whenever $k \in K_\gamma, \gamma = (k_1, \ldots, k_n; i) \in \Sigma$. We denote by $Alg(\Sigma)$ the class of all Σ-algebras.

Defining a superposition operation on sets of Σ-terms, we obtains a many-sorted algebra which satisfies some identities but in general not the superassociative law. This many-sorted algebra is called *I-clone* Λ. The satisfaction of a Σ-equation of each sort by a many-sorted algebra defines a Galois connection between the class of all many-sorted algebras and the set of all Σ-equations of the same sort. Σ-hypersubstitutions of each sort are mappings sending operation symbols to Σ-terms of the corresponding arities and sorts. Using such Σ-hypersubstitutions we define M-Σ-hyperidentities. The satisfaction of an M-Σ-hyperidentity of each sort by a many-sorted algebra defines a second Galois connection between the class of all many-sorted algebras and the set of all Σ-equations of the same sort. A class of many-sorted algebras is said to be M-solid if every Σ-equation for each sort which is satisfied as a Σ-identity is satisfied as an M-Σ-hyperidentity. On the basis of these two Galois connections we construct a conjugate pair of additive closure operators and are able to characterize M-solid varieties of many-sorted algebras.

We now consider the power set of sets of Σ-terms. Similarly, the satisfaction of a set of Σ-equations of the same sort by a class of many-sorted algebras defines a Galois connection between the power set of the class of all many-sorted algebras and the power set of the set of all Σ-equations of the same sort. Non-deterministic Σ-hypersubstitutions of each sort are mappings sending operation symbols to sets of Σ-terms of the corresponding arities and sorts. Using such non-deterministic Σ-hypersubstitutions we define non-deterministic Σ-hyperidentities. Satisfaction of an Σ-hyperidentity by a set of many-sorted algebras defines a second Galois connection between the power set of the class of all many-sorted algebras and the power set of the set of all Σ-equations of the same sort. A subset of the power set of the class of all many-sorted algebras is said to be non-deterministic solid if every set of Σ-equations of the same sort which is satisfied as a non-deterministic Σ-identity is satisfied as a non-deterministic Σ-hyperidentity. We are also able to characterize non-deterministic solid varieties of many-sorted algebras by using the theory of conjugate pairs of additive closure operators.

Finally we consider an application to tree languages. For some monoids of many-sorted Σ-hypersubstitutions we extend this correspondence to M-solid pseudovarieties of many-sorted algebras and M-solid varieties of many-sorted tree languages.