On aksiomatizability of a class of $d - MP$-modules

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d–MP–rings were introduced by H. Gorman [1] as differential rings for which the radical of every differential ideal is differential. Maximal among differential ideals of d–MP–rings are prime. d–MP–ring properties we re further studied by A. Nowicki [2]. W. Keigher [3] introduced so called special differential rings and proved they are equivalent to d–MP–rings. The concept of d–MP–module evolved quite naturally from the concept of d–MP–ring.

Unless otherwise specified, all rings are assumed to be associative with nonzero identity, and all modules are unitary left modules. By ideal we always mean a two-sided ideal. The term *differential ring* will refer to a ring R endowed with the set $\Delta = \{\delta_1, \delta_2, \ldots, \delta_n\}$ of n pairwise commutative ring derivations $\delta_i \colon R \to R$. In what follows, M denotes a left *differential R-module*; the differential structure on M is defined by the set $D = \{d_1, d_2, \ldots, d_n\}$ of pairwise commutative module derivations $d_i \colon M \to M$, consistent with the corresponding ring derivations. Assume that at least one derivation of Δ and D is nontrivial.

For $r \in R$, $m \in M$ we use the following notations:
$r^{(i_1,\ldots,i_n)} = (\delta_1^{i_1} \circ \ldots \circ \delta_n^{i_n})(r)$, $m^{(i_1,\ldots,i_n)} = (d_1^{i_1} \circ \ldots \circ d_n^{i_n})(m)$, $r^{(\infty)} = \{r^{(i_1,\ldots,i_n)} | i_1, i_2, \ldots, i_n \in \mathbb{N} \cup \{0\}\}$, $m^{(\infty)} = \{m^{(i_1,\ldots,i_n)} | i_1, i_2, \ldots, i_n \in \mathbb{N} \cup \{0\}\}$.

Let $[r]$ be the least differential ideal containing $r \in R$, and let $[m]$ be the least differential submodule containing $m \in M$. Note that $[r] = (r^{(\infty)})$, $[m] = (m^{(\infty)})$.

If X is an arbitrary subset of the differential R–module M, denote

$$X_{\#} = \{x \in M |\, x^{(i_1,i_2,\ldots,i_n)} \in X, \quad \forall i_1, i_2, \ldots, i_n \in \mathbb{N} \cup \{0\}\}.$$

The operator $()_{\#}$ has the following properties: $X_{\#}$ is differentially closed for any subset X of M; the union and the intersection of any family of differentially closed subsets is differentially closed; finite products and sums of differentially closed subsets are differentially closed; images and preimages of differentially closed subsets under differential homomorphisms are differentially closed; for any subset X of D-module M, $X_{\#}$ is the largest differentially closed subset of M contained in X.

A differential R–module M is called *d–MP–module* if for any prime differential submodule N of M the submodule $N_{\#}$ is a prime differential submodules of M.

For a d–MP–module M the following conditions are equivalent [6]:

1. any quasi-prime submodule N of M is prime;
2. any quasi-prime submodule N of M is radical, i. e. $\mathrm{rad}(N) = N$;
3. any prime submodule, minimal over some differential submodule, is differential;
4. radical of each differential submodule is a differential submodule.

It is therefore easy to see that in a d–MP–module maximal among differential submodules are prime, which explains the term d–MP.

A differential R–module M is called *differentially prime* [6] if $\mathrm{Ann}_l(N) = \mathrm{Ann}_l(M)$ for every nonzero differential submodule N of M. A differential submodule N of M is called *differentially prime* [6] if M/N is differentially prime.

Let S be a dm–system of R. A non-empty subset $S^\star$ of the differential module M over R is called an *Sdm–system* of the module M [6] if for any $s \in S$ and $x \in X$ there exist $r \in R$ and $i_1, i_2, \ldots, i_n \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$ such that $srx^{(i_1,i_2,\ldots,i_n)} \in S^\star$. If all the module derivations are trivial, we obtain the notion of an Sm–system of a module over non-commutative ring. For a regular differential module the above concept transforms into dm–system, which is introduced in [4].

A differential submodule $\mathcal{N}$ of a differential module M is differentially prime if and only if $M \backslash \mathcal{N}$ is an Sdm–system of M for some dm–system S of the ring R. A differential submodule $\mathcal{P}$ of the differential module M is differentially prime if and only if $IN \subseteq \mathcal{P}$ follows $N \subseteq \mathcal{P}$ or $I \subseteq \mathrm{Ann}_l(M/\mathcal{P})$ for every differential ideal I and ev ery differential submodule N of M. The latter condition is equivalent to $[r][m] \subseteq \mathcal{P}$ follows $m \in \mathcal{P}$ or $r \in \mathrm{Ann}_l(M/\mathcal{P})$ for any $r \in R$ and $m \in M$. (see [6])

A differential submodule $\mathcal{Q}$ of the differential R–module M is called *quasi-prime* if there exists an Sm–system $S^\star$ of M such that $\mathcal{Q}$ is a maximal among the differential submodules of M not meeting $S^\star$.

Theorem 1. *A differential R–module M is a d–MP–module if and only of its localization $M_\mathfrak{p}$ is a d–MP–module for every prime differential ideal $\mathfrak{p}$ of the ring R.*

Theorem 2. *If every differentially prime submodule of the differential module M is prime, then the module M is a d–MP–module.*

Let I be an infinite set, and let $\mathcal{U}$ be a nonprincipal ultrafilter over I. Suppose that for each $i \in I$ N_i is a submodule of the differential R_i–module M_i, then one can construct a submodule of $\prod_{i\in I} N_i/\mathcal{U}$ of the ultraproduct $\prod_{i\in I} M_i/\mathcal{U}$, which is an ultraproduct of a family of submodules $\{N_i\}_{i\in I}$ with respect to $\mathcal{U}$ [4].

Theorem 3. *The ultraproduct of any family of $d - MP$–modules with respect to the nonprincipal ultrafilter is a $d - MP$–module.*

Theorem 4. *The class of $d - MP$–modules is axiomatizable.*

References

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