Transitive modes

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Let us recall that a mode is an idempotent and medial algebra [1],[2]. In this work we study the connection between right (left) invertible medial algebras and right (left) group of binary operations. As a consequence, we prove the structure theorem for transitive modes and get a new characterization of multiplicative groups of fields (Mal'tsev problem).

The set of all binary operations on Q is denoted by $\mathcal{F}_Q^2$,and we consider the following two binary operations on this set:

$$A \cdot B(x, y) = A(x, B(x, y)),$$
$$A \circ B(x, y) = A(B(x, y), y),$$

where $A, B \in \mathcal{F}_Q^2$, $x, y \in Q$. These operations are called right and left multiplication of binary operations and they were studied in works of various authors [3]–[9].

The set $\mathcal{F}_Q^2$ forms a monoid under the right (left) multiplication of binary operations. The mapping, $A \to A^*$, is an isomorphism between semigroups $\mathcal{F}_Q^2(\cdot)$ and $\mathcal{F}_Q^2(\circ)$, where $A^*(x, y) = A(y, x)$. The operation, $E(x, y) = y$ ($F(x, y) = x$), is an identity element for the semigroup, $\mathcal{F}_Q^2(\cdot)$ (for $\mathcal{F}_Q^2(\circ)$).

The binary operation $A \in \mathcal{F}_Q^2$ is right (left) invertible, if the equation $A(a, x) = b$ ($A(y, a) = b$) has the unique solution $x \in Q$ ($y \in Q$). The unique solutions x, y are usually denoted by $x = A^{-1}(a, b)$ and $y =^{-1} A(b, a)$. Hence, $A \cdot A^{-1} = A^{-1} \cdot A = E$ for the right invertible operation A, and $^{-1}A \circ A = A \circ {}^{-1}A = F$ for the left invertible operation A.

For application of right (left) invertible operations in geometry or topology (knot theory) see [10], [11].

The set of all right (left) binary invertible operations on the set Q is denoted by $\mathcal{F}_Q^r$ (and $\mathcal{F}_Q^\ell$). The set $\mathcal{F}_Q^r$ ($\mathcal{F}_Q^\ell$) is a group under the right (left) multiplication of binary operations. These two groups are also isopmorphic.

The binary algebra $(Q; \Sigma)$ is called right (left) invertible if every operation $A \in \Sigma$ is right (left) invertible.

For $\Sigma \subseteq \mathcal{F}_Q^r$ we denote by (Σ) the subgroup of group $\mathcal{F}_Q^r(\cdot)$ generated by the subset Σ. If $\Sigma \subseteq \mathcal{F}_Q^\ell$, then by $((\Sigma))$ is denoted the subgroup of the group $\mathcal{F}_Q^\ell(\circ)$ generated by subset Σ.

Theorem 1. *If the right invertible algebra $(Q; \Sigma)$ is a medial (mode), then the extended algebra $(Q; (\Sigma))$ is also medial (mode).*

Theorem 2. *If the left invertible algebra $(Q; \Sigma)$ is medial (mode) then the extended algebra $(Q; ((\Sigma)))$ is also medial (mode).*

Following [6], we define the binary algebra $(Q;\Sigma)$, with $E, F \in \Sigma$, to be transitive, if the following conditions are valid:

a) $|Q| \geqslant 2$;

b) for every $a, b, c \in Q$, where $b \neq a \neq c$, there exists the operation $A \in \Sigma \setminus \{F\}$ such that $A(a, b) = c$;

c) $(Q; \Sigma \setminus \{F\})$ is a right invertible algebra.

Example 3. If $|Q| = 2$ and $\Sigma = \{E, F\}$, then $(Q;\Sigma)$ is a transitive mode.

Example 4. If $Q(\cdot)$ is a nontrivial abelian group and

$$\Sigma^0 = \{A_q |\, A_q(x, y) = y \cdot q,\ q \in Q,\ x, y \in Q\},$$

$\Sigma = \Sigma^0 \cup \{F\}$, then $(Q;\Sigma)$ is a medial transitive algebra, but $(Q;\Sigma)$ is not a mode.

Example 5. If $Q(+, \cdot)$ is a field with an identity element, $e \in Q$,

$$\Sigma = \{A_q |\, A_q(x, y) = qx + (e - q)y,\ q \in Q,\ x, y \in Q\},$$

or

$$\Sigma = \{A_q |\, A_q(x, y) = (e - q)x + qy,\ q \in Q,\ x, y \in Q\},$$

then $(Q;\Sigma)$ is a transitive mode.

Using the above-mentioned results, we prove a structure theorem for transitive modes.

As a consequence, we get a new characterization of multiplicative groups of fields.

References

[1] Romanowska A.B., Smith J.D.H., *Modes*, Singapore: World Scientific, 2002.

[2] Jezek J., Kepka T., *Medial groupoids*, Praha, 1983.

[3] Stein S.K., *On the foundation of quasigroups*, Trans. Amer. Math. Soc. 85(1957), 228-256.

[4] Urbanik K., *On algebraic operations in idempotent algebras*, Colloquium Math. XIII(1965), 129-157.

[5] Belousov V.D., *Systems of quasigroups with generalized identities*, Uspekhi. Mat. Nauk 20(1965), 75-146 [English transl. in Russian Math. Surveys 20(1965), 73-143].

[6] Movsisyan Yu.M., *Multiplicative group of a field and hyperidentities*, Izv. AN SSSR Ser. Mat. 53(5), 1989, 1040-1055 [English transl. in Math. USSR, Izvestiya 35(1990), 377-391].

[7] Movsisyan Yu.M., *Hyperidentities in algebras and varieties*, Uspekhi Mat. Nauk 53(1), 1998, 61-114 [English transl. in Russian Math. Surveys 53(1), 1998, 57-108].

[8] Movsisyan Yu.M., *Binary representations of algebras with at most two binary operations. A Cayley theorem for distributive lattices*, International Journal of Algebra and Computation, Vol. 19, N1(2009), 97-106.

[9] Bloom S.L., Esik Z., Manes E.G., *A cayley theorem for Boolean algebras*, Amer. Math. Monthly 97(1900), 831-833.

[10] Pambuccian V., *Euclidean geometry problems rephrased in terms of midpoints and point-reflections*, Elem. Math., 60(2005), 19-24.

[11] Matveev S.V., *Distributive groupoids in knot theory*, Mat. Sb. 119(1), 1982, 78-88 [English transl. in Mathematics of the USSR-Sbornik,47(1),1984].