Free modals

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Algebras considered in this talk are modes and modals. Such algebras were introduced and investigated in details by A. Romanowska and J.D.H. Smith ([2], [3]). *Modes* (M, Ω) are characterized by two basic properties. They are *idempotent*, in the sense that each singleton is a subalgebra, and *entropic*, i.e. any two of its operation commute. The two properties may also be expressed by means of identities:

$$\omega(x, \dots, x) \approx x, \qquad \text{(idempotent law)},$$
$$\omega(\phi(x_{11}, \dots, x_{n1}), \dots, \phi(x_{1m}, \dots, x_{nm})) \approx$$
$$\phi(\omega(x_{11}, \dots, x_{1m}), \dots, \omega(x_{n1}, \dots, x_{nm})), \qquad \text{(entropic law)},$$

for every n-ary $\omega \in \Omega$ and m-ary $\phi \in \Omega$.

An operation ω on a set A is said to distribute over a binary operation $+$ if and only if for any $1 \leq i \leq n$ and elements $x_1, \dots, x_i, y_i, \dots, x_n \in A$:

(1)
$$\omega(x_1, \dots, x_i + y_i, \dots, x_n) =$$
$$\omega(x_1, \dots, x_i, \dots, x_n) + \omega(x_1, \dots, y_i, \dots, x_n).$$

A *modal* is an algebra $(M, \Omega, +)$ such that (M, Ω) is a mode, $(M, +)$ is a (join) semilattice (with semilattice order $\leq$, i.e. $x \leq y \Leftrightarrow x+y = x$) and the operations $\omega \in \Omega$ distribute over $+$. The name "modal" was intended both to refer to the relationship with modes and to suggest the analogy with modules. Examples of modals include distributive lattices, dissemilattices - algebras $(M, \cdot, +)$ with two semilattice structures $(M, \cdot)$ and $(M, +)$ in which the operation $\cdot$ distributes over the operation $+$, and the algebra $(\mathbb{R}, \underline{I}^0, max)$ defined on the set of real numbers, where $\underline{I}^0$ is the set of the following binary operations: $\underline{p} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$; $(x, y) \mapsto (1-p)x + py$, for each $p \in (0, 1) \subset \mathbb{R}$.

Given a mode variety $\mathcal{V}$, a modal $(M, \Omega, +)$ is called $\mathcal{V}$-*modal* if its mode reduct (M, Ω) lies in $\mathcal{V}$. A. Romanowska and J.D.H. Smith ([2], [3]) described the free $\mathcal{V}$-modals in the case $\mathcal{V}$ is the variety of modes defined by linear identities (We call a term t *linear*, if every variable occurs in t at most once. An identity $t \approx u$ is called *linear*, if both terms t and u are linear.). The main aim of this talk is to describe the free object in arbitrary varieties of modals.

For a given set M let us denote by $\wp(M)$ the family of all non-empty subsets of M. For any n-ary operation $\omega : M^n \to M$ we define *the complex operation* $\omega : \wp(M)^n \to \wp(M)$ in the following way:

$$\omega(A_1, \dots, A_n) := \{\omega(a_1, \dots, a_n) \mid a_i \in A_i\},$$

where $\emptyset \neq A_1, \ldots, A_n \subseteq M$. *The power* (*complex* or *global*) *algebra* of an algebra (M, Ω) is the algebra $(\wp(M), \Omega)$.

As it was shown by A. Romanowska and J.D.H. Smith in [2], for a given mode (M, Ω), the sets MS of non-empty subalgebras and MP of finitely generated non-empty subalgebras of (M, Ω) have a mode structure under the ω-complex products satisfying each linear identity true in (M, Ω). Moreover, modes (MS, Ω) and (MP, Ω) are subalgebras of power algebra $(\wp(M), \Omega)$.

A. Romanowska and J.D.H. Smith also proved that for a given mode (M, Ω), sets MS and MP have an additional (join) semilattice structure $+$ obtained by setting

$$A_1 + A_2 := \langle A_1 \cup A_2 \rangle,$$

for any $A_1, A_2 \in MS$, where $\langle X \rangle$ denotes the subalgebra of (M, Ω) generated by the set X.

These two structures, mode and semilattice, are related by distributive laws (1). In this way, we obtain algebras $(MS, \Omega, +)$ and $(MP, \Omega, +)$ that provide basic examples of modals.

Let $\mathcal{V}$ be a variety of Ω-modes and let $\mathcal{M}_{\mathcal{V}}$ denote the variety of all $\mathcal{V}$-modals.

Theorem 1. *Let X be a set. For any variety $\mathcal{V}$ of modes, the modal $(F_{\mathcal{V}}(X)P, \Omega, +)$ of finitely generated non-empty subalgebras of the free $\mathcal{V}$-mode over X, has the universal mapping property for $\mathcal{M}_{\mathcal{V}}$ over X.*

Note that if $(F_{\mathcal{V}}(X)P, \Omega, +) \in \mathcal{M}_{\mathcal{V}}$, then it is, up to isomorphism, the unique algebra in $\mathcal{M}_{\mathcal{V}}$ with the universal mapping property freely generated by a set X.

Corollary 2. *Let $\mathcal{V}$ be a variety of Ω-modes. The modal $(F_{\mathcal{V}}(X)P, \Omega, +)$ is free over a set X in the variety $\mathcal{M}_{\mathcal{V}}$ if and only if $(F_{\mathcal{V}}(X)P, \Omega, +) \in \mathcal{M}_{\mathcal{V}}$.*

For a variety $\mathcal{V}$ of modes let $\mathcal{V}^*$ be its *linearization*, the idempotent variety defined by the linear identities satisfied in $\mathcal{V}$. Obviously, $\mathcal{V}^*$ is a variety of modes, and contains $\mathcal{V}$ as a subvariety.

It is well known (see for example [3]) that for any variety $\mathcal{V}$ of modes, the variety generated by the class $\{(AS, \Omega) \mid (A, \Omega) \in \mathcal{V}\}$ is included in $\mathcal{V}^*$. So by Corollary 2 we immediately obtain the following result proved earlier by A. Romanowska and J.D.H. Smith.

Theorem 3. [2] *Let $\mathcal{V}$ be a variety of modes. The modal $(F_{\mathcal{V}^*}(X)P, \Omega, +)$ is free over a set X in the variety $\mathcal{M}_{\mathcal{V}^*}$.*

Let $\mathcal{V}$ be a variety of Ω-modes, $\mathcal{M} \subseteq \mathcal{M}_{\mathcal{V}}$ a (non-trivial) variety of modals and X be a set. Define the congruence

$$\Theta(X) := \bigcap \{\phi \in Con(F_{\mathcal{V}}(X)P, \Omega, +) \mid (F_{\mathcal{V}}(X)P, \Omega, +)/\phi \in \mathcal{M}\}.$$

Theorem 4. *The algebra $(F_{\mathcal{V}}(X)P, \Omega, +)/\Theta(X)$ is free over a set X in $\mathcal{M}$.*

Corollary 5. *If the algebra $(F_{\mathcal{V}}(X)P, \Omega, +) \in \mathcal{M}$, then it is free in $\mathcal{M}$ over a set X.*

Let $(M, \Omega, +)$ be a modal generated by a set $X \subseteq M$. Denote by $\langle X \rangle_\Omega$ the subalgebra of Ω-reduct (M, Ω) generated by the set X. The algebra $(\langle X \rangle_\Omega, \Omega)$ is necessarily a submode of (M, Ω) and contains all elements from $(M, \Omega, +)$ obtained as results of term operations of Ω only. We will call it the *full Ω-mode subreduct of a modal* $(M, \Omega, +)$.

Let $\mathcal{MV}$ denote the class of all modals for which there exists a set of generators such that their full Ω-mode subreduct lies in $\mathcal{V}$.

It is well known that the subreducts of algebras in a given quasivariety again form a quasivariety. (See [1].) Let $\mathcal{Q}$ be a quasivariety of Γ-algebras of a given type $\tau : \Gamma \to \mathbb{N}$. Let $\mathfrak{B}\Gamma$ be a set of *derived operations* or *term operations* of Γ and let $\Omega \subseteq \mathfrak{B}\Gamma$. Consider a quasivariety $\mathcal{Q}_\Omega$ of Ω-algebras isomorphic to Ω-subreducts of $\mathcal{Q}$-algebras.

Theorem 6. *The free $\mathcal{Q}_\Omega$-algebra $(F_{\mathcal{Q}_\Omega}(X), \Omega)$ over X is isomorphic to the Ω-subreduct $\langle X \rangle_\Omega$, generated by X, of the free $\mathcal{Q}$-algebra $(F_\mathcal{Q}(X), \Gamma)$.*

Free algebras in a quasivariety $\mathcal{Q}_\Omega$ are also free in the variety $V(\mathcal{Q}_\Omega)$ generated by $\mathcal{Q}_\Omega$. (see [1])

As a corollary one obtains characterization of free algebras in a quasivariety of Ω-subreducts of modals in a given variety of modals.

Corollary 7. *Let $\mathcal{M}$ be a variety of modals. Let $\mathcal{M}_\Omega$ be a quasivariety of Ω-subreducts of modals in $\mathcal{M}$. Then the free $\mathcal{M}_\Omega$ - mode $(F_{\mathcal{M}_\Omega}(X), \Omega)$ over X is isomorphic to the full Ω-subreduct $\langle X \rangle_\Omega$ of the free $\mathcal{M}$-modal $(F_\mathcal{M}(X), \Omega, +)$.*

Note that even if we have a free modal $(F_\mathcal{M}(X), \Omega, +)$ in given variety $\mathcal{M} \subseteq \mathcal{MV}$, its full Ω-subreduct $(\langle X \rangle_\Omega, \Omega)$ needn't be a free algebra in $\mathcal{V}$.

Corollary 8. *For any variety $\mathcal{V}$ of modes,*

$$\mathcal{M}_\mathcal{V} \subseteq \mathcal{MV} \subseteq \mathcal{M}_{\mathcal{V}^*}.$$

In particular, $\mathcal{MV}^* = \mathcal{M}_{\mathcal{V}^*}$.

In general, the class $\mathcal{MV}$ is not a variety.

Theorem 9. *Let $\mathcal{V}$ be a variety of Ω-modes. Each modal $(M, \Omega, +) \in \mathcal{MV}$ generated by a set X is a homomorphic image of $(F_\mathcal{V}(X)P, \Omega, +)$.*

References

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[2] Romanowska A. B., Smith J. D. H.: *Modal Theory*, Heldermann Verlag, Berlin, 1985.
[3] Romanowska A. B., Smith J. D. H.: *Modes*, World Scientific, Singapore, 2002.