On the Impicit Algebraic Geometry on the Categories of Universal Algebras

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Let $\mathscr{K}$ be some class of universal algebras of some fixed signature σ and let $\vec{\mathscr{K}}$ be a category, which objects are $\mathscr{K}$-algebras and morphisms are (either all or some part of) homomorphisms of some $\mathscr{K}$-algebras into the others. Let us recall the definition of n-ary implicit operation f on the category $\vec{\mathscr{K}}$: it is the family of n-ary functions $f_{\mathscr{A}}$ on the basic sets of $\mathscr{K}$-algebras $\mathscr{A}$ which commutes with any $\mathscr{K}$-morphisms. As $\mathrm{IF}_n(\vec{\mathscr{K}})$ we denote the family of all n-ary implicit operations on the category $\vec{\mathscr{K}}$.

As $\mathscr{M} = \mathscr{M}(\mathscr{K})$ we denote the variety generated by the class $\mathscr{K}$, as $\mathscr{F}_{\mathscr{M}}(Y)$ — the $\mathscr{M}$-free Y-generated algebra. As $\mathscr{F}^{\vec{\mathscr{K}}}_{\mathscr{M}}(n)$ (for $n \in \omega$) we denote the algebra $\mathscr{F}_{\mathscr{M}}(\{x_1, \ldots, x_n\} \cup \mathrm{IF}_n(\vec{\mathscr{K}}))$. For any $\mathscr{K}$-algebra $\mathscr{A} = \langle A; \sigma \rangle$ it is natural to identity the cortege $\vec{a} = \langle a_1, \ldots, a_n \rangle$ elements from $\mathscr{A}$ with the homomorphism $\varphi_{\vec{a}}$ of the algebra $\mathscr{F}^{\vec{\mathscr{K}}}_{\mathscr{M}}(n)$ in the algebra $\mathscr{A}$ ($\varphi_{\bar{a}}(x_i) = a_i$, $\varphi_{\bar{a}}(f(x_1, \ldots, x_n)) = f_{\mathscr{A}}(\varphi_{\bar{a}}(x_1), \ldots, \varphi_{\bar{a}}(x_n))$, for $f \in \mathrm{IF}_n(\vec{\mathscr{K}})$).

Following the standart methods in algebraic geometry of universal algebras (for example as in [1]) we construct the Galya-connection between the subsets of the space A^n and the family of the binary relations on the algebra $F^{\vec{\mathscr{K}}}_{\mathscr{M}}(n)$ and we introduce the concepts:
$\vec{\mathscr{K}}$-implicit algebraic subsets of the algebra $\mathscr{A}$ and $\mathscr{A}-$
$\vec{\mathscr{K}}$-closed congruence of the algebra $\mathscr{F}^{\vec{\mathscr{K}}}_{\mathscr{M}}(n)$ (the family of this congruences we denote as $Con^{\vec{\mathscr{K}},A}_{\mathscr{A}} \mathscr{F}^{\vec{\mathscr{K}}(n)}_{\mathscr{M}}$).

Let $\vec{\mathscr{K}}$-implicit equation for the category $\vec{\mathscr{K}}$ be any formal equation of the two $\vec{\mathscr{K}}$-implicit operations. The set of the solutions of this $\vec{\mathscr{K}}$-implicit equation in $\mathscr{K}$-algebra $\mathscr{A}$ can be defined naturally. The formula

$$\underset{i \in \mathrm{I}}{\&} f^i(x_1, \ldots, x_n) = g^i(x_1, \ldots, x_n) \to h(x_1, \ldots, x_n) = k(x_1, \ldots, x_n)$$

$$\text{if } f^i, g^i, h, k \in \mathrm{IF}_n(\vec{\mathscr{K}})$$

naturally defines the concept of $\vec{\mathscr{K}}$-implicit ∞-quasiidentity. By analogy with the relation $\leq^A$ (see [2]) of the comparison of the algebraic geometries of $\mathscr{K}$-algebras $\mathscr{A}_1$ and $\mathscr{A}_2$, we can define the relation of quasiorder $\leq^{A,\vec{\mathscr{K}}}$ (the comparison of the $\vec{\mathscr{K}}$-implicit algebraic geometries of $\mathscr{K}$-algebras) on the class $\mathscr{K}$.

$$\mathscr{A}_1 \leq^{A,\vec{\mathscr{K}}} \mathscr{A}_2 \iff Con^{\vec{\mathscr{K}},A}_{\mathscr{A}_2} \mathscr{F}^{\vec{\mathscr{K}}}_{\mathscr{M}}(n) \subseteq Con^{\vec{\mathscr{K}},A}_{\mathscr{A}_1} \mathscr{F}^{\vec{K}}_{\mathscr{M}}(n).$$

So, we have the following

THEOREM. For any cathegory $\vec{\mathscr{K}}$ of universal algebras and any $\mathscr{K}$-algebras $\mathscr{A}_1$, $\mathscr{A}_2$ the relation $\mathscr{A}_1 \leq^{A,\vec{\mathscr{K}}} \mathscr{A}_2$ holds if and only if any $\vec{\mathscr{K}}$-implicit ∞-quasiidentety which is valid on the algebra $\mathscr{A}_1$ is also valid on the algebra $\mathscr{A}$.

This theorem implies several corollaries for different categories of algebras:

(1) when $\mathscr{K}$ is a variety of universal algebras and $\vec{\mathscr{K}}$-morphisms is class of all homomorphisms of $\mathscr{K}$-algebras into $\mathscr{K}$-algebras;
(2) when $\mathscr{K}$ is a pseudovariety of finite algebras and $\vec{\mathscr{K}}$-morphisms is a class of all homomorphisms of $\mathscr{K}$-algebras into $\mathscr{K}$-algebras;
(3) when $\mathscr{K}$ is a $\forall$-class of algebras and $\vec{\mathscr{K}}$-morphisms is a class of all embeddings of some $\mathscr{K}$-algebras into $\mathscr{K}$-algebras.

For example the following takes place.

COROLLARY. For any $\forall$-class $\mathscr{K}$ of universal algebras the relation $\mathscr{A}_1 \leq^{A,\vec{\mathscr{K}}} \mathscr{A}_2$ is valid iff any finitely generated non-one-element subalgebra of the algebra $\mathscr{A}_2$ can be isomorphically embedded into the algebra $\mathscr{A}_1$.

References

[1] B.Plotkin. Some notions of algebraic geometry in universal algebra.- Algebra and Analisis, v.9, No.4, 1997, p.224-248.

[2] A.G.Pinus. The geometrical scales of the varieties of algebras and quasiidentities.- Math. works, v.12, No.2, p.160-169.