On the coset structure of categorical skew lattices

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The study of noncommutative lattices began in 1949 with Pascual Jordan's paper [6] that was motivated by certain questions in quantum mechanics. Skew lattices turned out to be the most fruitful class of noncommutative lattices and were thus studied the most. The foundations of the modern theory of skew lattices can be found in Jonathan Leech's 1989 paper [8].

A *skew lattice* **S** is a set S equipped with two associative binary operations $\vee$ and $\wedge$, called the *meet* and the *join*, that satisfy the absorption laws $(b \wedge a) \vee a = a = a \vee (a \wedge b)$ and their duals. Both $\wedge$ and $\vee$ are idempotent which follows directly from the absorption laws.

Already in Leech's first paper [8] a special attention was devoted to the skew lattices in rings. The operations defined by $x \wedge y = xy$ and $x \vee y = x + y - xy$ succeeded to provide a rather large class of examples which have motivated many of the properties studied in the general case. A good survey on skew lattices can be found in Leech [7].

On a given skew lattice **S** the *natural partial order* $\leq$ is defined by $x \leq y$ iff $x \wedge y = x = y \wedge x$ or dually $x \vee y = y = y \vee x$. On the other hand, the relation $\mathcal{D}$ is defined by $x\mathcal{D}y$ iff $x \vee y \vee x = x$ and $y \vee x \vee y = y$ or dually, $x \wedge y \wedge x = x$ and $y \wedge x \wedge y = y$. $\mathcal{D}$ is called the *natural equivalence* and it coincides with Green's relation $\mathcal{D}$ on both semigroups $(S; \wedge)$ and $(S; \vee)$. Within skew lattices it is a congruence. By Leech's First Decomposition Theorem every skew lattices is a lattice of its $\mathcal{D}$-classes. A pair of comparable $\mathcal{D}$-classes induce partitions on each $\mathcal{D}$-class in the pair, and the blocks of these partitions are called *cosets*.

Theorem 1 (Leech [10])**.** *In a skew lattice* **S** *with comparable $\mathcal{D}$-classes $A > B$, B is partitioned by the cosets of A in B; dually A is partitioned by the cosets of B in A. The image of any element $a \in A$ in B is a transversal of the cosets of A in B; dual remarks hold for any $b \in B$ and the cosets of B in A. Moreover, given cosets $B \vee a \vee B$ in A and $A \wedge b \wedge A$ in B a natural bijection of cosets is given by: $x \in B \vee a \vee B$ corresponds to $y \in A \wedge b \wedge A$ if and only if $x \geq y$. The operations $\wedge$ and $\vee$ on $A \cup B$ are determined jointly by the coset bijections and the rectangular structure of each $\mathcal{D}$-class.*

Altrough the studdy of Skew Lattices in Rings one wants to know what happens within *skew chains*, that is skew lattices who's quocient with the congruence $\mathcal{D}$ is a chain. One of the interesting properties here is the *categoricity*. The n ame comes directly from the definition as categorical skew lattices are the ones for whom coset bijections form a category. Already in 1993, Leech defines categorical skew lattices within his geometric perspective on skew lattices under [10]. In thi s work, categoricity is characterized as follows:

Theorem 2. *Let S be a skew lattice. Given distinct $\mathcal{D}$-classes $A > B > C$ in a categorical skew lattice with elements $a \in A$, $b \in B$ and $c \in C$ that satisfy $a > b > c$, consider the coset bijection $\varphi_{B,C} : C \vee b \vee C \to B \wedge c \wedge B$ from B to C that takes b to c and the coset bijection $\psi_{B,A} : A \wedge b \wedge A$ from B to A that takes b to a. S is categorical iff the restrictions $\overline{\varphi}_{B,C} : (A \wedge b \wedge A) \cap (C \vee b \vee C) \to A \wedge c \wedge A$ and $\overline{\psi}_{B,A} : (A \wedge b \wedge A) \cap (C \vee b \vee C) \to C \vee a \vee C$ are bijections.*

Cvetko-Vah's paper [4] within skew lattices in rings gives us a hint on how the coset laws that describe this property might look like. Stated in the variety of skew lattices in rings in [9], this coset laws are generalized onto skew latt ices. On recent times the study of Categorical Skew Lattices as revealed its importance. In this work we present several characterizations of this property as well as the description of its coset structure by the the following theorem

Theorem 3. *Let $\mathbf{S}$ be a skew lattice. The following statements are equivalent:*

(i) *$\mathbf{S}$ is categorical,*

(ii) *given any skew chain $\{A > B > C\}$ in $\mathbf{S}$, $a \in A, b \in B, c \in C$ and any $x, x' \in C$, $A \wedge x \wedge A = A \wedge x' \wedge A$ if and only if $B \wedge x \wedge B = B \wedge x' \wedge B$ and, exist $b \in B$ such that*

$$A \wedge (x \vee b \vee x) \wedge A = A \wedge (x' \vee b \vee x') \wedge A.$$

(iii) *given any skew chain $\{A > B > C\}$ in $\mathbf{S}$, $a \in A, b \in B, c \in C$ and any $x, x' \in A$, $C \vee x \vee C = C \vee x' \vee C$ if and only if $B \vee x \vee B = B \vee x' \vee B$ and, exist $b \in B$ such that*

$$C \vee (x \wedge b \wedge x) \vee A = A \vee (x' \wedge b \wedge x') \vee A.$$

Example 4. Let F be a field with characteristic different from 2, $n \in \mathbb{N}$ and $\mathbf{S}$ a right handed skew lattice in $M_n(F)$. If $\mathbf{S}$ has two comparable $\mathcal{D}$-classes $A > B$ then given $a \in A$ and $b \in B$, $bA = \{ba : a \in A\}$ is the coset of A in B and $B \circ a = \{b + a - ba : b \in B\}$ is the coset of B in A.

The standard form for right handed skew lattices in $M_n(F)$ was described in [4], based on the standard form for pure bands in matrix rings that was developed by Fillmore at al. in [2] and [3]. It is described as follows. Let A and B be a non-comparable $\mathcal{D}$-classes of a skew lattice $\mathbf{S}$ with the meet class M and the join class J. Then, given any matrices $m \in M$, $j \in J$, $a \in A$ and $b \in B$, a basis for F^n exists such that in this basis they have the followin block forms :

$$m = \begin{bmatrix} I & m_{12} & m_{13} & m_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, a = \begin{bmatrix} I & 0 & a_{13} & a_{14} \\ 0 & I & 0 & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$b = \begin{bmatrix} I & b_{12} & 0 & b_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & b_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } j = \begin{bmatrix} I & 0 & 0 & j_{14} \\ 0 & I & 0 & j_{24} \\ 0 & 0 & I & j_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

To see the Theorem 3 coset laws let

$$m = \begin{bmatrix} I & x & y & z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad a = \begin{bmatrix} I & 0 & w & u \\ 0 & I & 0 & v \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus

$$(m\nabla a)j_0 = \begin{bmatrix} I & 0 & y & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

On the other hand,

$$mj_0 = \begin{bmatrix} I & x & y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad ma_0 = \begin{bmatrix} I & x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore,

$$mJ = m'J \text{ iff for all } a \in A,\ (m\nabla a)J = (m'\nabla a)J \text{ and } mA = m'A.$$

Dually, we obtain

$$M\nabla j = M\nabla j' \text{ iff for all } a \in A,\ M\nabla(aj) = M\nabla(aj') \text{ and } A\nabla j = A\nabla j'.$$

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