Representation Theorem for Rough Set Lattices Determined by Quasiorders

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Rough sets are introduced by Z. Pawlak in [7]. In rough theory it is assumed that our knowledge about a universe of discourse U is given by a binary relation reflecting distinguishability or indistinguishability of the elements of U. Originally, Pawlak assumed that this binary relation is an equivalence, but in the literature can be found numerous studies in which approximations are determined by also other types of relations than just equivalences.

If R is an arbitrary binary relation on U, then for any subset $X \subseteq U$, the *lower approximation* of X is defined by

$$X^{\blacktriangledown} = \{x \in U \mid R(x) \subseteq X\}$$

and the *upper approximation* of X is

$$X^{\blacktriangle} = \{x \in U \mid R(x) \cap X \neq \emptyset\},$$

where $R(x) = \{y \in U \mid x\,R\,y\}$. The *rough set* of X is the pair $\mathcal{A}(X) = (X^{\blacktriangledown}, X^{\blacktriangle})$ and the set of all rough sets is

$$RS = \{\mathcal{A}(X) \mid X \subseteq U\}.$$

The set RS may be canonically ordered by the coordinatewise order:

$$(X^{\blacktriangledown}, X^{\blacktriangle}) \leq (Y^{\blacktriangledown}, Y^{\blacktriangle}) \iff X^{\blacktriangledown} \subseteq Y^{\blacktriangledown} \text{ and } X^{\blacktriangle} \subseteq Y^{\blacktriangle}.$$

In case R is an equivalence relation, the structure of RS is well-known [1], [2], [3], [6], [8]. Then, RS is with respect to the order $\leq$ a regular double Stone lattice isomorphic to $\mathbf{2}^I \times \mathbf{3}^J$, where $\mathbf{2}$ and $\mathbf{3}$ are the chains of two and three elements, I is the set of singleton R-classes, J is the set of non-singleton equivalence classes of R, $\mathbf{2}^I$ is the pointwise ordered set of all mappings from I to the two-element chain, and $\mathbf{3}^J$ is the pointwise ordered set of all maps from J to the 3-element chain. In addition, RS forms a three-valued Łukasiewicz algebra. If R is reflexive and symmetric or just transitive, then RS is not always even a semilattice. If R is symmetric and transitive, then the structure of RS is as in case of equivalences [4].

In [5] we proved that if U if a non-empty set and R is a quasiorder on U, then RS is a complete sublattice of $\wp(U) \times \wp(U)$, where $\wp(U)$ denotes the set of all subsets of U. This then means that RS is a completely distributive complete lattice such that

$$\bigwedge_{i \in I} \mathcal{A}(X_i) = \Big(\bigcap_{i \in I} X_i^{\blacktriangledown}, \bigcap_{i \in I} X_i^{\blacktriangle}\Big) \quad \text{and} \quad \bigvee_{i \in I} \mathcal{A}(X_i) = \Big(\bigcup_{i \in I} X_i^{\blacktriangledown}, \bigcup_{i \in I} X_i^{\blacktriangle}\Big)$$

for all $\{\mathcal{A}(X_i) \mid i \in I\} \subseteq RS$. We also proved that the mapping

$$c\colon RS \to RS,\ \mathcal{A}(X) \mapsto \mathcal{A}(U \setminus X)$$

is a de Morgan complement, and therefore the algebra

$$(RS, \vee, \wedge, c, (\emptyset, \emptyset), (U, U))$$

is a de Morgan algebra.

To rough sets lattices determined by equivalences, there exists the following representation theorem: for every regular double Stone algebra A, there exists a set U and an equivalence R on U such that A is isomorphic to a subalgebra of RS. The main objective of this presentation is to show the following representation theorem.

Theorem 1. *Let $(A, \vee, \wedge, c, 0, 1)$ be a Nelson algebra such that $(A, \vee, \wedge)$ is an algebraic lattice. Then, there exists a universe U and a quasiorder R on U such that A and RS are isomorphic Nelson algebras.*

By applying the above theorem, we may also prove the following representation theorem for rough set lattices determined by equivalences.

Corollary 2. *Let $(A, \vee, \wedge, c, 0, 1)$ be a semi-simple Nelson algebra such that $(A, \vee, \wedge)$ is an algebraic lattice. Then, there exists a universe U and an equivalence R on U such that A and RS are isomorphic Nelson algebras.*

References

[1] Stephen D. Comer, *An algebraic approach to the approximation of information*, Fundamenta Informaticae **14** (1991), 492–502.

[2] Stphane P. Demri and Ewa S. Orlowska, *Incomplete information: Structure, inference, complexity*, Monographs in Theoretical Computer Science. An EATCS Series, Springer-Verlag, Berlin, Heidelberg, 2002.

[3] Mai Gehrke and Elbert Walker, *On the structure of rough sets*, Bulletin of Polish Academy of Sciences. Mathematics **40** (1992), 235–245.

[4] Jouni Järvinen, *The ordered set of rough sets*, Lecture Notes in Computer Science **3066** (2004), 49–58.

[5] Jouni Järvinen, Sándor Radeleczki, and Laura Veres, *Rough sets determined by quasiorders*, Order **26** (2009), 337–355.

[6] Piero Pagliani and Mihir Chakraborty, *A geometry of approximation. Rough set theory: Logic, algebra and topology of conceptual patterns*, Springer, 2008.

[7] Zdzisław Pawlak, *Rough sets*, International Journal of Computer and Information Sciences **11** (1982), 341–356.

[8] Jacek Pomykała and Janusz A. Pomykała, *The Stone algebra of rough sets*, Bulletin of Polish Academy of Sciences. Mathematics **36** (1988), 495–512.