On mode reducts of semimodules

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Modes are idempotent and entropic algebras. More precisely, an algebra (A, Ω) of type $\tau : \Omega \longrightarrow \mathbb{Z}^+$ is called a *mode* if it is *idempotent* and *entropic*, i.e. each singleton in A is a subalgebra and each operation $\omega \in \Omega$ is actually a homomorphism from an appropriate power of the algebra. Both properties can also be expressed by the following identities:

(I) $\forall \omega \in \Omega, \;\; x \dots x\omega = x$

(E) $\forall \omega, \varphi \in \Omega$, with m-ary ω and n-ary φ,

$$\begin{aligned}&(x_{11} \dots x_{1m}\omega) \dots (x_{n1} \dots x_{nm}\omega)\varphi \\ &= (x_{11} \dots x_{n1}\varphi) \dots (x_{1m} \dots x_{nm}\varphi)\omega,\end{aligned}$$

satisfied in the algebra (A, Ω).

Examples of modes are provided by

• normal bands (idempotent and entropic semigroups),

• many binary (or groupoid) modes appearing in combinatorics and geometry,

• *affine spaces* (or *affine modules*), algebras equivalent to full idempotent reducts of modules over commutative (unital) rings, and their reducts and subreducts (subalgebras of reducts,

• *semi-affine spaces* (or *affine semimodules*), algebras equivalent to full idempotent reducts of (unital zero-preserving) semimodules over (unital) commutative semirings (with absorbing zero), and their reducts and subreducts.

Commutative unital semirings, and semimodules over them, are defined in similar fashion as for commutative rings and modules over them, abelian groups being replaced by commutative monoids. Moreover, the zero o of each semiring S is assumed to be an absorbing zero, i.e. $xo = o$ for each $x \in S$, and each semimodule is assumed to be zero-preserving, i.e. $xo = 0$, where 0 is the zero of the monoid reduct. All semirings and semimodules considered here are of this type.

A long-standing problem in the theory of modes asked for a characterization of those modes that embed as subreducts into semimodules over commutative unital semirings.

The embeddability problem has now been solved as follows.

Theorem 1 (M. Stronkowski, 06, D. Stanovský, 09)**.** *A mode embeds as a subreduct into a semimodule over a commutative semiring if and only if it satisfies the so-called Szendrei identities.*

For a given type τ, Szendrei identities arise from each word (term) of type τ of the form

$$x_{11} \dots x_{1n}w \, \dots \, x_{1n} \dots x_{nn}w \, w,$$

where w is a derived operator with n variables defining a basic operation of the reduct in question, by interchanging x_{ij} and x_{ji} for fixed $1 \leq i, j \leq n$.

Among the classes of embeddable modes known before let us mention the following ones:

• groupoid modes (or idempotent medial groupoids) (J. Ježek, T. Kepka);

• cancelative modes (A. Romanowska, J. D. H. Smith);

• certain sums of cancellative modes (A. Romanowska, J. D. H. Smith, A. Zamojska-Dzienio);

• semilattice modes (K. Kearnes).

It is thus clear that each class of modes of a given type is divided into two subclasses, the class of modes embeddable into semimodules, and the class of non-embeddable modes. The first class contains two important subclasses. One consists of affine spaces over commutative (unital) rings, while the other consists of semi-affine spaces over (unital) commutative semirings (with absorbing zero).

The class $\underline{\underline{R}}$ of affine spaces over a fixed commutative ring R (or affine R-spaces) is known to be a variety. Such varieties are characterized as varieties of Mal'cev modes. No comparable characterization of semi-affine spaces over a fixed commutative semiring is currently known. The first difficulty one encounters is the following: While the full idempotent reduct of a module over a commutative ring is always non-trivial, the idempotent reducts of semimodules in question may actually be trivial. For example, any free semimodule over the semiring of natural numbers has as non-trivial derived (or "term") operations the linear combinations $x_1n_1 + \cdots + x_kn_k$. The only such idempotent operations are projections. This raises the following:

Problem 2. Characterize semimodules over commutative semirings with non-trivial idempotent reducts.

As idempotent reducts of such semimodules are in fact mode reducts, a solution to this problem would provide a characterization of all semimodules with non-trivial semi-affine space reducts, and a description of semi-affine spaces containing all embeddable modes as subreducts.

While we are still not able to solve this problem in full generality, we can provide some partial solutions.

First recall that, by Płonka theory for regularised varieties, it follows that for certain (very general) varieties $\mathcal{V}$ of algebras, the regularization $\widetilde{\mathcal{V}}$ of $\mathcal{V}$ and the class of Płonka sums of $\mathcal{V}$-algebras coincides. (There are two versions of this theorem: for algebras without constant operations, and for algebras with constants.) In particular, this is true for idempotent varieties and for varieties of modules.

Let $\widetilde{\mathcal{MOD}}_R$ be the regularization of the variety $\mathcal{MOD}_R$ of R-modules, modules over a commutative unital ring R. Let $\underline{\underline{R}}$ be the variety of affine R-spaces, and $\widetilde{\underline{\underline{R}}}$ its regularization. The Płonka sums in $\widetilde{\underline{\underline{R}}}$ of affine R-spaces over semilattices with a smallest element are called *bounded* Płonka sums. The full idempotent reducts of algebras in $\widetilde{\mathcal{MOD}}_R$ are called *semi-affine R-spaces.*

Theorem 3. [1] *The subclass* $\widetilde{\underline{\underline{R}}}^b$ *of bounded Płonka sums of the regularization* $\widetilde{\underline{\underline{R}}}$*, consists precisely of algebras term-equivalent to semi-affine R-spaces.*

Let $\mathcal{SMOD}_{R^0}$ be the variety of semimodules over the semiring obtained from a commutative unital ring R by adding a new zero to the ring R.

Proposition 4. *The varieties* $\widetilde{\mathcal{MOD}}_R$ *and* $\mathcal{SMOD}_{R^0}$ *are equivalent.*

Theorem 5. [1] *Let R be a commutative unital ring. Then the class of idempotent reducts of semimodules in* $\mathcal{SMOD}_{R^0}$ *coincides with the class* $\underline{\underline{\widetilde{R}}}^b$.

Note that the Mal'cev operation P of affine R-spaces becomes the so-called *regularized Mal'cev* operation (or *regMal'cev* operation) in Płonka sums of affine R-spaces. A mode with a regMal'cev operation is called a *regMal'cev mode.* A characterization of semi-affine R-spaces is then completed by the following theorem.

Theorem 6. [1] *Each regMal'cev mode that is a bounded Płonka sum of affine R-spaces is equivalent to a semi-affine R-space.*

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