Semigroup Homomorphisms and Idempotent Elements in Sets of Tree Languages

Klaus Denecke
e-mail: kdenecke@rz.uni-potsdam.de
University of Potsdam, Germany

Napaporn Sarasit*
e-mail: napaporn_sarasit@hotmail.com
University of Potsdam,Germany

S.L. Wismath
e-mail: ---
University of Lethbridge, Canada

Tree languages are sets of terms. We consider an indexed set of operation symbols f_i, where f_i is n_i-ary for every $i \in I$, and a finite alphabet $X_n := \{x_1, \ldots, x_n\}$. Sometimes we need a countably infinite alphabet $X := \{x_1, \ldots, x_n, \ldots\}$. Let $\tau := (n_i)_{i \in I}, n_i \geq 1$ be the sequence of all arities of the operation symbols f_i. In many applications the restriction to finite types is justified. The sequence τ is called the type of the terms. Then the set $W_\tau(X_n)$ of all n-ary terms of type τ is inductively defined in the usual way. Let $W_\tau(X)$ be the set of all terms of type τ. We denote by $\mathcal{F}_\tau(X_n) = (W_\tau(X_n); (\overline{f_i})_{i\in I})$ the absolutely free algebra of type τ generated by the n-element alphabet X_n. To each term of type τ there corresponds a tree. Therefore instead of terms w e will also speak of trees. Any element of the power set $\mathcal{P}(W_\tau(X_n))$ is called a tree language of type τ.

On the power set $\mathcal{P}(W_\tau(X_n))$ of the set $W_\tau(X_n)$ of all n-ary terms of type τ we define a binary associative operation $+_n$. For $B_1, B_2 \subseteq W_\tau(X_n)$ we set

$$B_1 +_n B_2 := \hat{S}^n(B_1, B_2, \ldots, B_2).$$

This binary operation is derived from the superposition operation

$$\hat{S}^n_m : \mathcal{P}(W_\tau(X_n)) \times (\mathcal{P}(W_\tau(X_m)))^n \to \mathcal{P}(W_\tau(X_m))$$

on sets of terms and preserves recognizability. We study the properties of sets of n-ary terms which are idempotent with respect to this operation. We obtain that every idempotent element L of $(\mathcal{P}(W_\tau(X_n)); +)$
forms a subalgebra of $(W_\tau(X_n); S^n)$ with $L \cap X_n \neq \emptyset$. Next we consider varieties of tree languages consisting of idempotent elements only. Let VL be a variety of tree languages with $L \cap X_n \neq \emptyset$ fo r every $L \in VL_n$ and every $n \geq 1$. Then for every $L \in VL_n \setminus \{W_\tau(X_n)\}$ and for every $n \geq 1$ there exists a homomorphism $h_L : \mathcal{F}_\tau(X_n) \to \mathcal{F}_\tau(X_n)$ such that $(h_L)^{-1}(L) = \overline{L}$ wh ere $\overline{L}$ is the complement of L. This means that if we let VL be an idempotent variety (with respect to $+_n$) of tree languages of type τ, then for every $L \in VL_n \setminus \{W_\tau(X_n)\}$ and for every $n \geq 1$ there exists a h omomorphism $h_L : \mathcal{F}_\tau(X_n) \to \mathcal{F}_\tau(X_n)$ such that $(h_L)^{-1}(L) = \overline{L}$. One more consequence is: If $L \in VL_n$, then for every $\tilde{L} \subseteq L$ with $|\tilde{L}| \leq n$ the universe of $\langle \tilde{L} \rangle_{\mathcal{F}_\tau(X_n)}$ is a subset of L where $\langle \tilde{L} \rangle_{\mathcal{F}_\tau(X_n)}$ for a subset $\tilde{L} \subseteq W_\tau(x_n)$ is the subalgebra of $\mathcal{F}_\tau(X_n)$ which is generated by $\tilde{L}$. Moreover we propose to extend the variety theory of tree languages to other

classes of tree languages. Finally, we study and compare properties of semigroup homomorphisms $\varphi : (\mathcal{P}(W_\tau(X_n)); +_n) \to (\mathcal{P}(W_\tau(X_m)); +_m)$ with properties of homomorphisms $\overline{h} : \mathcal{F}_\tau(X_n) \to \mathcal{F}_\tau(X_m)$ between the absoulutely free algebras. If there is a finite image under a semigroup homomorphism φ, then any set of variables is mapped to a set of variables. Varieties of tree languages are closed under inverses of homomorphisms

$$h : \mathcal{F}_\tau(X_n) \to \mathcal{F}_\tau(X_m).$$

If we are interested in languages $L \subseteq W_\tau(X_n)$ which are idempotent with respect to $+_n$ and which belong to varieties of tree languages, we should at first ask for connections between semigroup homomorphisms of $(\mathcal{P}(W_\tau(X_n)); +_n)$ and homomorphisms of the absolutely free algebra $\mathcal{F}_\tau(X_n)$. Any homomorphism $overlineh : \mathcal{F}_\tau(X_n) \to \mathcal{F}_\tau(X_m)$ is the uniquely determined extension of a mapping $h : X_n \to W_\tau(X_m)$. Any mapping $\overline{h} : \mathcal{F}_\tau(X_n) \to \mathcal{F}_\tau(X_m)$ induces a mapping $\hat{h} : \mathcal{P}(W_\tau(X_n)) \to \mathcal{P}(W_\tau(X_m)$ by $\hat{h}(L) := \{\overline{h}(a) \mid a \in L\}$. We want to call this "a mapping induced by h". By definition, $\hat{h}(L) = \overline{h}(L)$ for any $L \subseteq W_\tau(X_n)$. This implies that $\overline{h}$ is one-to-one or onto if and only if $\hat{h}$ has this property. But for the inverse mappings we have $(\overline{h})^{-1}(L) = \{a \in W_\tau(X_n) \mid \overline{h}(a) \in L\} \in \{B \subseteq W_\tau(X_n) \mid \hat{h}(B) = L\} = (\hat{h})^{-1}(L)$ and $B \subseteq (\overline{h})^{-1}(L)$ for any $L \subseteq W_\tau(X_n)$ and any $B \in (\hat{h})^{-1}(L)$.

The first problem is to find conditions for h under which $\hat{h}$ is a semigroup homomorphism. Since a semigroup homomorphism $\hat{h}$ maps idempotent elements of $(\mathcal{P}(W_\tau(X_n)); +_n)$ to idempotent elements of $(\mathcal{P}(W_\tau(X_m)); +_m)$, $\hat{h}$ must be induced by a mapping $h : X_n \to X_m$. Clearly, there are semigroup homomorphisms which are not induced by homomorphisms of the absolutely free algebras. As an example we consider the following mapping $\varphi : \mathcal{P}(W_\tau(X_n)) \to \mathcal{P}(W_\tau(X_m))$. Let $B \subseteq W_{tau}(X_m)$ be an infinite idempotent set with respect to $+_m$. Then we define $\varphi(A) = B$ for any $A \subseteq W_\tau(X_n)$ and have $\varphi(A_1 +_n A_2) = B = B +_m B = \varphi(A_1) +_n \varphi(A_2)$ for any $A_1, A_2 \subseteq W_\tau(X_n)$. T his shows that φ is a semigroup homomorphism, but is not induced by a mapping $h : X_n \to X_m$.