Weak Homomorphisms between Functorial Algebras

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In algebraic structure theory there are usually considered algebras of a fixed common signature and so the notion of a homomorphism is restricted to those situations where the corresponding algebras are of the same similarity type. However, this kind of re straint is rather unnecessary in a lot of investigations in which, as in the theory of completeness, the term operations of an examined algebra play the essential role.

As introduced by E. Marczewski in [7], a mapping $\varphi : A \to B$ is said to be a *weak homomorphism from* $\mathcal{A} = (A, F)$ *into* $\mathcal{B} = (B, G)$ if for each n-ary fundamental operation $f \in F$ $(n \in \mathbb{N})$ there exists an n -ary term operation g of $\mathcal{B}$ such that

$$\varphi(f(a_1, \dots, a_n)) = g(\varphi(a_1), \dots, \varphi(a_n))$$

holds for all $(a_1, \dots, a_n) \in A^n$ and vice versa, for every n-ary fundamental operation $g \in G$ $(n \in \mathbb{N})$ there is an n-ary term operation f of $\mathcal{A}$ satisfying the same condition. Those weak homo morphisms, in particular weak endo- and automorphisms, were investigated under various aspects, especially by K. Głazek. (For more details see [1]-[5].)

Furthermore, there exists a well-known generalization of universal algebras where the types are endofunctors of the category of sets. So, whenever F is a set functor, an F-algebra is an ordered pair $\mathcal{A} = (A, \alpha)$ consisting of some set A and a map $\alpha : F(A) \to A$. However, the existence of a homomorphism between those algebras necessitates a common type of the concerned structures.

In my talk, I want to develop the notion of a weak homomorphism between those more general structures without restricting the sope of considerations by the necessity of a common type. Concerning an F_1-algebra $\mathcal{A} = (A, \alpha)$ and an F_2-algebra $\mathcal{B} = (B, \beta)$, a map $\varphi : A \to B$ is said to be a *weak homomorphism from* $\mathcal{A}$ *into* $\mathcal{B}$ if on the carrier $Q := \varphi[A]$ there exist an F_1-algebra $\mathcal{Q}_1 = (Q, \gamma_1)$ and a n F_2-algebra $\mathcal{Q}_2 = (Q, \gamma_2)$ such that the following conditions are satisfied:

(i) The structures $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are *algebraically equivalent*, i.e. for each set I the direct products $\mathcal{Q}_1^I$ and $\mathcal{Q}_2^I$ have exactly the same closed subsets.

(ii) $\dot{\varphi} : \mathcal{A} \to \mathcal{Q}_1$ and $\subseteq_Q^B : \mathcal{Q}_2 \to \mathcal{B}$ are homomorphisms, i.e. the diagram

$$\begin{tikzcd}
F_1(A) \arrow[r, "F_1\dot{\varphi}"] \arrow[d, "\alpha"'] & F_1(Q) \arrow[dr, dashed, "\gamma_1"] & & F_2(Q) \arrow[r, "F_2(\subseteq_Q^B)"] \arrow[dl, dashed, "\gamma_2"'] & F_2(B) \arrow[d, "\beta"] \\
A \arrow[rr, two heads, "\dot{\varphi}"] \arrow[rrrr, bend right, "\varphi"'] & & Q \arrow[rr, hook, "\subseteq_Q^B"] & & B
\end{tikzcd}$$

commutes.

In the case of universal algebras, this new definition coincides with E. Marczewski's concept (cf. [9]).

In [9] it was shown that if the involved functors weakly preserve kernels, the composition of two weak homomoprhisms is a weak homomorphism. Moreover, in many respects weak homomorphisms behave like usual homomorphisms. For instance, it turns out that kern els of weak homomorphisms are congruence relations, weakly homomorphic images and preimages of subalgebras, respectively, are subalgebras, and that there exist certain cancellation properties, too. These nice results substantiate that the introduced concep t of weak homomorphisms between differently typed functorial algebras is indeed a useful and promising idea.

For a class $\mathcal{K}$ of set functors which weakly preserve kernels, it is a natural idea to investigate the category $\mathbf{Set}_{\mathcal{K}}$ consisting of all algebras of types from $\mathcal{K}$ as objects and all weak homomorphisms between them as morphisms. And it is still an open problem to find general conditions for $\mathcal{K}$ under which $\mathbf{Set}_{\mathcal{K}}$ has products. In my talk, I want to discuss two very suggesting constructions where the canonical projections become weak h omomorphisms. However, in general the constructed algebras fail to fulfill the universal product property.

At the end, I would like to talk about another critical point of the topic: Throughout the work with weak homomorphisms, the axiom of choice plays an important role. This has to be expected since the definition of a weak homomorphism tells something about infinite direct products of algebras. However, it is possible to replace the assumption of the axiom of choice by requiring an additional property of the functors under consideration. Assuming that the concerned functors *strongly preserve epimorphism s*, we do not need the axiom of choice to prove the obtained results. But then the axiom of choice is significant for the quantity of this restricted class of functors.

References

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