Existence varieties of complemented modular lattices

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The talk is intended to present some structural results on existence varieties of complemented modular lattices and also those of (von Neumann) regular rings.

All the classes of structures we consider here are *abstract*; that is, they are closed under isomorphic copies. As usual, S, H, P, and P_u denote the operators of taking substructures, homomorphic images, Cartesian products, and ultraproducts, respectively.

All the lattices which we consider here have a least element, and we consider them in the signature $\sigma = \{0, \cdot, +\}$. In particular, by a sublattice, we always mean a 0-sublattice. A lattice L is *sectionally complemented*, if it satisfies the following first-order sentence φ_l:

$$\forall x \forall y \exists z\ [xyz = 0]\ \wedge\ [xy + z = x].$$

A lattice L with a greatest element 1 is *complemented*, if for any $x \in L$, there is $y \in L$ such that $xy = 0$ and $x + y = 1$.

Let Λ be a commutative ring with unit. We consider Λ-algebras in the signature $\sigma = \{0, \Lambda, -, +, \cdot\}$ and we consider rings in the signature $\sigma = \{0, -, +, \cdot\}$, cf. [4]. We emphasize that rings do not necessarily have a unit. An algebra (a ring, respectively) is (*von Neumann*) *regular*, if it satisfies the following first-order sentence φ_r:

$$\forall x \exists y\ xyx = x.$$

In a regular ring R, all finitely generated (right) ideals in R are principal and are generated by an idempotent. Of course, one can view regular rings as regular $\mathbb{Z}$-algebras with right ideals being subalgebras.

By $\mathbb{L}(R)$ we denote the lattice of principal (right) ideals. Then $\mathbb{L}(R)$ is a sectionally complemented modular lattice. Moreover, R is Artinian if and only if $\mathbb{L}(R)$ is of finite height; in this case, R necessarily has a unit and $\mathbb{L}(R)$ has a greatest element.

For a (right) module M, $\mathbb{L}(M)$ denotes the lattice of all submodules of M and $\mathrm{End}(M)$ denotes the endomorphism ring of M. If V_D is a vector space over a division ring, then $\mathbb{L}(V_D)$ is a subdirectly irreducible Arguesian lattice. If V_D is finite dimensional, then $\mathbb{L}(V_D)$ is simple. For a division ring D and for a positive integer n, D^n_D denotes an n-dimensional vector space over D, and $D^{n\times n}$ denotes the ring of $n \times n$ matrices over D (sometimes viewed as an Λ-algebra for a suitable commutative ring Λ); in particular, the ring $D^{n\times n}$ is simple regular Artinian and it is isomorphic to the endomorphism ring $\mathrm{End}(D^n_D)$ of D^n_D. Moreover, for any division ring D and any positive integer n, $\mathbb{L}(D^n_D) \cong \mathbb{L}(D^{n\times n})$ and $\mathbb{L}(D^n_D)$ is of height n.

The next statement captures the classical view on coordinatization of projective spaces due to Hilbert, Veblen and Young, Birkhoff, and von Neumann.

Theorem 1. *Let L be a simple Arguesian complemented lattice of height n, $3 \leqslant n < \omega$. Then $L \cong \mathbb{L}(R^{n\times n})$ for a division ring R. Moreover, this division ring R is unique up to isomorphism.*

Let Σ consist of axioms for modular lattices plus the sentence φ_l in case of lattices and let Σ consist of axioms for algebras (rings) plus the sentence φ_r in case of algebras (rings, respectively). For any $\mathcal{K} \subseteq \mathrm{Mod}(\Sigma)$, the operator $\mathsf{S}_\exists$ is defined by:

$$\mathsf{S}_\exists(\mathcal{K}) = \mathsf{S}(\mathcal{K}) \cap \mathrm{Mod}(\Sigma).$$

A class $\mathcal{K} \subseteq \mathrm{Mod}(\Sigma)$ is an *existence variety* or an *∃-variety*, if it is closed under H, P, and $\mathsf{S}_\exists$. In particular, any existence variety is closed under ultraproducts and elementary substructures, whence it forms an axiomatizable class.

The following statement gives structural information about existence varieties.

Theorem 2. [4] *For any class $\mathcal{K} \subseteq \mathrm{Mod}(\Sigma)$, the following statements hold:*

(1) the class $\mathsf{V}_\exists(\mathcal{K}) = \mathsf{HS}_\exists\mathsf{P}(\mathcal{K})$ is the smallest existence variety containing $\mathcal{K}$;
(2) any subdirectly $\mathsf{V}_\exists(\mathcal{K})$-irreducible algebra belongs to $\mathsf{HS}_\exists\mathsf{P}_u(\mathcal{K})$;
(3) any existence variety is generated by its finitely generated subdirectly irreducible algebras.

The following two statements provide more information on generators of existence varieties, cf. Theorem 2(3).

Theorem 3. [4]

(1) Any existence variety of sectionally complemented modular lattices is generated by its subdirectly irreducible finite height members.
(2) Any existence variety of Λ-algebras is generated by its subdirectly irreducible Artinian members.

Theorem 4. [4]

(1) The existence variety of all sectionally complemented modular lattices is generated by lattices of the form $\mathbb{L}\big(\mathbb{F}_p^n\big)$, where $n < \omega$ and $\mathbb{F}_p$ is a finite prime field of characteristic p.
(2) The existence variety of all regular rings is generated by matrix rings $\mathbb{F}_p^{n\times n}$, where $n < \omega$ and $\mathbb{F}_p$ is a finite prime field of characteristic p.

An explicit description of finite height members of existence varieties of sectionally complemented Arguesian lattices, as well as one of Artinian members of existence varieties of regular rings, is given in [5].

Corollary 5. [4]

(1) The equational theory of sectionally complemented modular lattices (with sectional complementation as a fundamental operation) is decidable.
(2) The equational theory of regular rings (with quasi-inversion as a fundamental operation) is decidable.