Equational quasigroup definitions

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We give some new equational definitions of quasigroups. Some other conditions when a groupoid is a quasigroup can be found in [5], [10], [9], [12], [15].

The subject of this paper is close with the subject of articles [15], [16]. We shall use basic terms and concepts from books [1], [3], [2], [13], [17], [7], [8].

Garrett Birkhoff in his famous book [4] defined equational quasigroup as an algebra with three binary operations $(Q, \cdot, /, \backslash)$ that fulfils the following six identities

$$x \cdot (x \backslash y) = y \tag{1}$$

$$(y/x) \cdot x = y \tag{2}$$

$$x \backslash (x \cdot y) = y \tag{3}$$

$$(y \cdot x)/x = y \tag{4}$$

$$x/(y \backslash x) = y \tag{5}$$

$$(x/y) \backslash x = y \tag{6}$$

We remark in [[14], page 11] the following problem is raised.

"Research properties of algebra $(Q, \cdot, /, \backslash)$ with various combinations of identities (1)–(6)."

Algebras with identities (1)–(6) are discussed in [15], [16]. For example the following "equivalence" theorem is proved.

Theorem 1. *(1) A groupoid $(Q, \cdot)$ is a left division groupoid if and only if there exists a left cancellation groupoid $(Q, \backslash)$ such that in the algebra $(Q, \cdot, \backslash)$ identity (1) is fulfilled.*

(2) A groupoid $(Q, \cdot)$ is a right division groupoid if and only if there exists a right cancellation groupoid $(Q, /)$ such that in the algebra $(Q, \cdot, /)$ identity (2) is fulfilled.

(3) A groupoid $(Q, \cdot)$ is a left cancellation groupoid if and only if there exists a left division groupoid $(Q, \backslash)$ such that in the algebra $(Q, \cdot, \backslash)$ identity (3) is fulfilled.

(4) *A groupoid* $(Q, \cdot)$ *is a right cancellation groupoid if and only if there exists a right division groupoid* $(Q, /)$ *such that in the algebra* $(Q, \cdot, /)$ *identity (4) is fulfilled.*

Information on properties of algebras with various sets of identities that are taken from identities (1)–(4) it is possible to deduce from Theorem 1.

It is well known the following

Lemma 2. *In algebra* $(Q, \cdot, \backslash, /)$ *with identities (1), (2), (3), (4) identities (5) and (6) are true* [17], [14], [15].

Therefore it is used the following T. Evans' equational definition of a quasigroup [6].

Definition 3. An algebra $(Q, \cdot, \backslash, /)$ with identities (1), (2), (3) and (4) is called a *quasigroup* [6], [4], [1], [3], [13], [5], [17].

Lemma 4. (1) *In algebra* $(Q, \cdot, \backslash, /)$ *identity (3) follows from identities (4) and (6).*

(2) *In algebra* $(Q, \cdot, \backslash, /)$ *identity (1) follows from identities (2) and (5).*

(3) *In algebra* $(Q, \cdot, \backslash, /)$ *identity (2) follows from identities (1) and (6).*

(4) *In algebra* $(Q, \cdot, \backslash, /)$ *identity (4) follows from identities (3) and (5).*

Example 5. (1) Define binary groupoid $(\mathbb{Z}, *)$ $x * y = x + 2y$ for all $x, y \in \mathbb{Z}$, where $(\mathbb{Z}, +, \cdot)$ is the ring of integers. Groupoid $(\mathbb{Z}, *)$ satisfies identities (2), (3), (4), (6) and it is not a quasigroup.

(2) Define binary groupoid $(\mathbb{Z}, *)$ $x * y = 2x + y$ for all $x, y \in \mathbb{Z}$, where $(\mathbb{Z}, +, \cdot)$ is the ring of integers. Groupoid $(\mathbb{Z}, *)$ satisfies identities (1), (3), (4), (5) and it is not a quasigroup.

(3) Define binary groupoid $(\mathbb{Z}, *)$ $x * y = x + [y/2]$ for all $x, y \in \mathbb{Z}$, where $(\mathbb{Z}, +, \cdot)$ is the ring of integers, $[y/2] = a$, if $y = 2 \cdot a$; $[y/2] = a$, if $y = 2 \cdot a + 1$. Groupoid $(\mathbb{Z}, *)$ satisfies identities (1), (2), (4), (5) and it is not a quasigroup.

(4) Define binary groupoid $(\mathbb{Z}, *)$ $x * y = [x/2] + y$ for all $x, y \in \mathbb{Z}$, where $(\mathbb{Z}, +, \cdot)$ is the ring of integers, $[x/2] = a$, if $x = 2 \cdot a$; $[x/2] = a$, if $xy = 2 \cdot a + 1$. Groupoid $(\mathbb{Z}, *)$ satisfies identities (1), (2), (3), (6) and it is not a quasigroup.

Theorem 6. (1) *An algebra* $(Q, \cdot, \backslash, /)$ *with identities (2), (3), (5) is a quasigroup.*

(2) *An algebra* $(Q, \cdot, \backslash, /)$ *with identities (1), (4), (6) is a quasigroup.*

In the following corollary we give definitions of equational quasigroup using four identities from the identities (1)–(6).

Corollary 7. (1) *An algebra* $(Q, \cdot, \backslash, /)$ *with identities (1), (2), (3), (5) is a quasigroup.*

(2) *An algebra* $(Q, \cdot, \backslash, /)$ *with identities (2), (3), (4), (5) is a quasigroup.*

(3) *An algebra* $(Q, \cdot, \backslash, /)$ *with identities (1), (2), (4), (6) is a quasigroup.*

(4) *An algebra* $(Q, \cdot, \backslash, /)$ *with identities (1), (3), (4), (6) is a quasigroup.*

(5) *An algebra* $(Q, \cdot, \backslash, /)$ *with identities (1), (3), (5), (6) is a quasigroup.*

(6) An algebra $(Q, \cdot, \backslash, /)$ with identities (1), (4), (5), (6) is a quasigroup.
(7) An algebra $(Q, \cdot, \backslash, /)$ with identities (2), (3), (5), (6) is a quasigroup.
(8) An algebra $(Q, \cdot, \backslash, /)$ with identities (2), (4), (5), (6) is a quasigroup.

Corollary 8. *An algebra $(Q, \cdot, \backslash, /)$ with any five identities from identities (1)–(6) is a quasigroup.*

The proof of Lemma 4 is obtained using Prover 9 [11].

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