Automorphisms of universal algebras

Jonathan D. H. Smith
e-mail: jdhsmith@iastate.edu
Iowa State University, Ames, USA

One of the standard exercises in universal algebra considers the set of automorphisms of a given algebra. This set of automorphisms forms a group under composition. The intention of the current work is to study the set $\mathrm{Aut}(A, \mathcal{C})$ of all automorphisms of all those algebras on a fixed set A that lie in a certain abstract class $\mathcal{C}$ of algebras. This global approach to automorphisms has recently received increasing attention in the theory of quasigroups (compare e.g. [2], [4]).

Taking $\mathcal{C}$ as an abstract class means that the set $\mathrm{Aut}(A, \mathcal{C})$ is a union of conjugacy classes in the group $A!$ of all permutations of the set A. For a finite set A, each such conjugacy class is specified by its cycle type, the partition of the size $|A|$ of A given by the lengths of the orbits on A under the action of a member of the conjugacy class. Thus one of the goals of the program is to specify, for a given abstract class $\mathcal{C}$ and for each natural number n, a list of the partitions of n that arise as cycle types of automorphisms of $\mathcal{C}$-algebras of order n. Since this set of cycle types will be empty if and only if n does not lie in the finite spectrum of $\mathcal{C}$, the specification o f the automorphism cycle types subsumes the specification of the finite spectrum of the given class $\mathcal{C}$. Tables 1–3 present sample lists of automorphism cycle types for GF(2)-spaces, abelian groups, and quasigroups , for orders up to 8. It is apparent that decreasing structure (e.g., fewer identities or fewer derived operations) progressively admits new cycle types.

n	Cycle types of n-element GF(2)-space automorphisms
1	1^1
2	1^2
4	$1^4,\ 2^1 1^2,\ 3^1 1^1$
8	$1^8,\ 2^2 1^4,\ 2^3 1^2,\ 3^2 1^2,\ 4^1 1^4,\ 7^1 1^1$

Table 1. GF(2)-space automorphisms

For a given abstract class $\mathcal{C}$ and cardinality n, a *representative set* of $\mathcal{C}$-algebras of order n have the property that their automorphisms include all the possible automorphism cycle types of $\mathcal{C}$-algebras of order n. If $\mathcal{C}$ is categorical in power n, there will be a singleton representative set. In general, the unique member of a singleton representative set is known as a *representative algebra*. The varieties of semilattices and lattices have a representative algebra in each (finite) cardinality.

n	Cycle types of n-element abelian group automorphisms
1	1^1
2	1^2
3	$1^3,\ 2^1 1^1$
4	$1^4,\ 2^1 1^2,\ 3^1 1^1$
5	$1^5,\ 2^2 1^1,\ 4^1 1^1$
6	$1^6,\ 2^2 1^2$
7	$1^7,\ 2^3 1^1,\ 3^2 1^1,\ 6^1 1^1$
8	$1^8,\ 2^2 1^4,\ 2^3 1^2,\ 3^2 1^2,\ 4^1 1^4,\ 7^1 1^1$

Table 2. Abelian group automorphisms

n	Cycle types of n-element quasigroup automorphisms
1	1^1
2	1^2
3	$1^3,\ 2^1 1^1,\ 3^1$
4	$1^4,\ 2^1 1^2,\ 2^2,\ 3^1 1^1$
5	$1^5,\ 2^2 1^1,\ 3^1 1^2,\ 4^1 1^1,\ 5^1$
6	$1^6,\ 2^2 1^2,\ 3^1 1^3,\ 3^2,\ 4^1 1^2,\ 5^1 1^1$
7	$1^7,\ 2^2 1^3,\ 2^3 1^1,\ 3^2 1^1,\ 4^1 1^3,\ 4^1 2^1 1^1,\ 5^1 1^2,\ 6^1 1^1,\ 7^1$
8	$1^8,\ 2^2 1^4,\ 2^3 1^2,\ 2^4,\ 3^2 1^2,\ 4^1 1^4,\ 4^1 2^1 1^2,\ 4^1 2^2,\ 4^2,\ 5^1 1^3,\ 6^1 1^2,\ 7^1 1^1$

Table 3. Quasigroup automorphisms

In general, it may be difficult to determine whether a given abstract class has a representative algebra of a given finite order.

While the set of automorphisms of a single algebra form a group under composition, the question of the algebraic structure of the full sets $\mathrm{Aut}(A, \mathcal{C})$ is not so easily resolved. The structure of a λ-ring [1], [3], [5], the *automorphism type ring* $\mathrm{ATR}_n(\mathcal{C})$, may be associated with $\mathrm{Aut}(A, \mathcal{C})$ for each abstract class $\mathcal{C}$ and set A of finite order n. These λ-rings are commutative, unital rings or $\mathbb{Z}$-algebras with additional unary operations, the lambda operations λ^k defined for each natural number k. In the ring of integers, $\lambda^k(l)$ is the binomial coefficient $\binom{l}{k}$. If the underlying ring is actually a $\mathbb{Q}$-algebra, then an alternative axiomatization is offered by the unary Adams operations ψ^k for positive integers k. The automorphism type rings are actually $\mathbb{C}$-algebras, whose complex dimensions agree with the total number of automorphism cycle types. The Adams operations then give vestiges of the positive powers in automorphism groups. The structure of the automorphism type rings is described by the following theorem.

Theorem 1. *Let $\mathcal{C}$ be an abstract class of algebras. Let A be a finite set of positive cardinality n. Let $R = \{(A, \Omega_1, \rho_1), \dots, (A, \Omega_d, \rho_d)\}$ be a minimal representative set of $\mathcal{C}$-algebras on A. For $1 \le i \le d$, let r_i be the order of $\mathrm{Aut}(A, \Omega_i, \rho_i)$, and let l_i be the cardinality of its set of $A!$-fused conjugacy classes, which span a subalgebra $S(A, \Omega_i, \rho_i)$ of the class algebra $\mathrm{Z}\mathbb{C}\,\mathrm{Aut}(A, \Omega_i, \rho_i)$ of $\mathrm{Aut}(A, \Omega_i, \rho_i)$. Let $r = \mathrm{lcm}\{r_1, \dots, r_d\}$.*

(a) *The automorphism type ring $\mathrm{ATR}_n(\mathcal{C})$ is a subdirect product of the algebras $S(A, \Omega_i, \rho_i)$ for $1 \le i \le d$.*

(b) *The $\mathbb{C}$-algebra $\mathrm{ATR}_n(\mathcal{C})$ has dimension satisfying* $d - 1 + \max\{l_i \mid 1 \le i \le d\} \le \dim_{\mathbb{C}} \mathrm{ATR}_n(\mathcal{C}) \le 1 - d + \sum_{i=1}^{d} l_i$.

(c) *The λ-ring $\mathrm{ATR}_n(\mathcal{C})$ satisfies the identities $\psi^{k+r} = \psi^k$ for* $0 < k \in \mathbb{Z}$. □

References

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