Hypergraph functor and attachment[1]

Sergejs Solovjovs
e-mail: sergejs.solovjovs@lu.lv
University of Latvia, Riga, Latvia

Since the inception of the notion of fuzzy topology much attention was paid to the possible means of interaction between fuzzy and crisp topological settings, in order to see whether fuzzy topology was doing anything new. In particular, different functors relating the categories **Top** of topological spaces and L-**Top** of L-topological spaces (L being a suitable complete lattice) appeared in the literature, providing the desired machinery for comparing classical and fuzzy developments. One of the mo st important examples in the field is *hypergraph functor*. Initiated by R. Lowen [8] and E. S. Santos [13], the concept was studied by many researchers [4],[6],[7],[11], but still failed to gai n much prominence in the fuzzy community. The main reasons were, firstly, the lack of information on functorial properties of the hypergraph functor and, secondly, remarkable differences in its definition by various authors (cf., e.g., those of U. Höhl e [6] and S. E. Rodabaugh [11]).

There has been several attempts to amend the situation. The former of the above-mentioned deficiencies was partly removed by W. Kotzé, T. Kubiak [7] and U. Höhle [6] by considering the hypergraph functor from the categorical poin t of view. The second deficiency, however, appeared more resistant and was approached to only recently by C. Guido [5]. Motivated by the concept of *quasi-coincidence* [9] (which is an analogue of the *intersection* property for fuzzy sets), he introduced the notion of *attachment* on a complete lattice as follows.

Definition 1. An *attachment family*, or more simply an *attachment* on a complete lattice L is a family $\mathcal{A} = \{F_a \,|\, a \in L\}$ of subsets of L such that $F_\perp = \varnothing$, and for every $a \in L\backslash\{\perp\}$, F_a is a completely prime filter of L ($\bigvee S \in F_a$ implies $S \bigcap F_a \neq \varnothing$).

The new concept gave rise to a functor L-**Top** $\xrightarrow{(-)^*}$ **Top**, which appeared to have striking similarities with the hypergraph functor. In particular, a slight modification of the notion of attachment (*generalized attachment*) done by C . Guido gave $(-)^*$ the power to provide a common framework for many (if not all) approaches to the topic. It is important to notice, however, that the author does not consider any categorical property of his functor apart from that of being an embed ding. It is the purpose of this talk to develop the categorical aspects of the attachment theory and its relationship to the hypergraph functor.

Based on our current research on topological properties which could be used in an arbitrary variety of algebras, we introduce the notion of *variety-based attachment*, replacing complete lattices of [5] by algebras (in an obvious sense, as set s with operations).

[1] This research was supported by the University of Latvia under ESF Project No. 2009/0223/1DP/1.1.1.2.0/09/APIA/VIAA/008.

Definition 2. Let $\mathbf{A}$ be a variety of algebras and let $\mathbf{A} \xrightarrow{(-)^*} \mathbf{Set}^{op}$ be a functor such that $A^* = |A|$ ($| - |$ is the underlying set of $\mathbf{A}$). $\mathbf{AttA}$ is the category, whose objects are triples $F = (\Omega\, F, \Sigma\, F, \Vdash)$ (called $\Sigma\, F$*-attachments on* $\Omega\, F$), where $\Omega\, F$ and $\Sigma\, F$ are algebras, and $\Omega\, F \xrightarrow{\Vdash} \mathbf{A}(\Omega\, F, \Sigma\, F)$ is a map. Morphisms $F_1 \xrightarrow{f=(\Omega\, f, \Sigma\, f)} F_2$ are $\mathbf{A} \times \mathbf{A}$-morphisms $(\Omega\, F_1, \Sigma\, F_1) \xrightarrow{(\Omega\, f, \Sigma\, f)} (\Omega\, F_2, \Sigma\, F_2)$ such that for every $a_1 \in \Omega\, F_1$ and every $a_2 \in \Omega\, F_2$,

$$(\Vdash_2(a_2))(\Omega\, f(a_1)) = (\Sigma\, f \circ \Vdash_1((\Omega\, f)^{*op}(a_2)))(a_1),$$

with the composition and identities being those of $\mathbf{A} \times \mathbf{A}$.

The notion gives rise to a new category for topology, which is a proper super-category of the currently dominating one for topological structures in the fuzzy community, introduced by S. E. Rodabaugh [12].

On the next step, we use the notion of *variety-based topological system* (developed in [15],[19], and motivated by the concept of *topological system* of S. Vickers [20], already modified by various authors [1], cite51.II,[3],[10]) to provide a *variety-based hypergraph functor*, which incorporates the respective fixed-basis concepts of [8],[11] and partly those of [6],[7]. Moreover, the above-mentioned functor $(-)^*$ of [5] is also included in the framework. Restricting the setting to the fixed-basis case, we give the sufficient conditions for the new functor to be an embedding, construct a right adjoint to it and show a relation of the obtained adjunction to that provided by U. Höhle [6] for the particular case of *frames* (complete lattices with the property that finite meets distribute over arbitrary joins).

The results of the talk not only make the nature of hypergraph functor more transparent, but also clearly show that its definition and many of its properties depend not on a particular lattice-theoretic peculiarity of the respective underlying structure fo r fuzziness, but rather on its categorically-algebraic aspects. This crucial fact makes a significant contribution to the new approach to topological structures introduced by us recently under the name of *categorically-algebraic* (*catalg*) *topology* [14],[16],[17],[18]. The new theory is based on both category theory and universal algebra (relying more on the former) that is reflected in its name. The main advantage of the catalg setting is the possibility of uniting (almost) all approaches to (fuzzy) topology, currently developed in mathematics, under one roof, ultimately erasing the border between crisp and fuzzy developments and postulating the slogan: algebra is at the bottom of everything.

Acknowledgements

The results of this talk were obtained in collaboration with Prof. C. Guido (University of Salento, Italy) and his research team.

References

[1] S. Abramsky and S. Vickers, *Quantales, observational logic and process semantics*, Math. Struct. Comput. Sci. **3** (1993), 161–227.

[2] J. T. Denniston, A. Melton, and S. E. Rodabaugh, *Lattice-valued topological systems*, Abstracts of the 30th Linz Seminar on Fuzzy Set Theory (U. Bodenhofer, B. De Baets, E. P. Klement, and S. Saminger-Platz, eds.), Johannes Kepler Universität, Linz, 2009, pp. 24–31.

[3] J. T. Denniston, A. Melton, and S. E. Rodabaugh, *Lattice-valued predicate transformers and interchange systems*, Abstracts of the 31st Linz Seminar on Fuzzy Set Theory (P. Cintula, E. P. Klement, and L. N. Stout, eds.), Johannes Kepler Universität, Linz, 2010, pp. 31–40.

[4] G. Gerla, *Representations of fuzzy topologies*, Fuzzy Sets Syst. **11** (1983), 103–113.

[5] C. Guido, *Fuzzy points and attachment*, to appear in Fuzzy Sets Syst.

[6] U. Höhle, *A note on the hypergraph functor*, Fuzzy Sets Syst. **131** (2002), no. 3, 353–356.

[7] W. Kotzé and T. Kubiak, *Fuzzy topologies of Scott continuous functions and their relation to the hypergraph functor*, Quaest. Math. **15** (1992), no. 2, 175–187.

[8] R. Lowen, *A comparison of different compactness notions in fuzzy topological spaces*, J. Math. Anal. Appl. **64** (1978), 446–454.

[9] P.-M. Pu and Y.-M. Liu, *Fuzzy topology I: Neighborhood structure of a fuzzy point and Moore-Smith convergence*, J. Math. Anal. Appl. **76** (1980), 571–599.

[10] P. Resende, *Quantales, finite observations and strong bisimulation*, Theor. Comput. Sci. **254** (2001), no. 1-2, 95–149.

[11] S. E. Rodabaugh, *Point-set lattice-theoretic topology*, Fuzzy Sets Syst. **40** (1991), no. 2, 297–345.

[12] S. E. Rodabaugh, *Categorical Foundations of Variable-Basis Fuzzy Topology*, Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory (U. Höhle and S. E. Rodabaugh, eds.), The Handbooks of Fuzzy Sets Series, vol. 3, Dordrecht: Kluwer Academic Publishers, 1999, pp. 273–388.

[13] E. S. Santos, *Topology versus fuzzy topology*, Youngstown State University, 1977, preprint.

[14] S. Solovjovs, *Categorically-algebraic frameworks for Priestley representation theory*, submitted to Proc. of the 79th Workshop on General Algebra, February 12 - 14, 2010, Olomouc, Czech Republic.

[15] S. Solovjovs, *Embedding topology into algebra*, Abstracts of the 30th Linz Seminar on Fuzzy Set Theory (U. Bodenhofer, B. De Baets, E. P. Klement, and S. Saminger-Platz, eds.), Johannes Kepler Universität, Linz, 2009, pp. 106–110.

[16] S. Solovjovs, *Categorically-algebraic dualities*, Abstracts of Applications of Algebra XIV, Institute of Mathematics and Computer Science of Jan Długosz University, Częstochowa, 2010, pp. 45–48.

[17] S. Solovjovs, *Composite variety-based topological theories*, Abstracts of the 10th Conference on Fuzzy Set Theory and Applications (FSTA 2010) (E. P. Klement, M. Radko, P. Struk, and E. Drobná, eds.), Armed Forces Academy of General M. R. Štef 'anik in Liptovský Mikuláš, 2010, pp. 116–118.

[18] S. Solovjovs, *Powerset operator foundations for categorically-algebraic fuzzy sets theories*, Abstracts of the 31st Linz Seminar on Fuzzy Set Theory (P. Cintula, E. P. Klement, and L. N. Stout, eds.), Johannes Kepler Universität, Linz, 2010, pp. 143–151.

[19] S. Solovyov, *Variable-basis topological systems versus variable-basis topological spaces*, to appear in Soft Comput.

[20] S. Vickers, *Topology via Logic*, Cambridge University Press, 1989.