Centralizer clones are preserved by category equivalences

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A functor $F : C \to D$ is an equivalence functor from a category C to a category D if the following conditions are satisfied: 1) for each object $\mathbf{B} \in Obj(D)$, there exists an object $\mathbf{A} \in Obj(C)$ such that $F(\mathbf{A}) \cong \mathbf{B}$; 2) for any objects $\mathbf{A}, \mathbf{B} \in Obj(C)$, if $f_1, f_2 : \mathbf{A} \longrightarrow \mathbf{B}$ are morphisms and $F(f_1) = F(f_2)$ then $f_1 = f_2$; and 3) for any objects $\mathbf{A}, \mathbf{B} \in Obj(C)$, if $g : F(\mathbf{A}) \longrightarrow F(\mathbf{B})$ is a morphism then there exists a morphism $f : \mathbf{A} \longrightarrow \mathbf{B}$ such that $F(f) = g$. We say that two categories C and D are category equivalent if there is an equivalence functor $F : C \to D$.

Any variety can be regarded as a category where the objects are the algebras in the variety and the morphisms are the homomorphisms. Therefore one can ask which properties of a variety are preserved under category equivalence.

There are many properties which can be transfered among category equivalence of varieties. As a result of B. A. Davey and H. Werner [1], for an equivalence functor $F : V \to W$ from a variety V to a variety W, the following statements are fullfilled.

1) If $\mathbf{A} \in V$ is finitely generated or one-element or finite, then $F(\mathbf{A})$ has the same property.

2) F maps epimorphisms to epimorphisms and F maps monomorphisms to monomorphisms.

3) For each $\mathbf{A} \in V$, the subalgebra lattices of $\mathbf{A}$ and of $F(\mathbf{A})$ are isomorphic.

4) For each $\mathbf{A} \in V$, the congruence lattices of $\mathbf{A}$ and of $F(\mathbf{A})$ are isomorphic.

5) The subvarieties lattice of V and of W are isomorphic.

6) F induces s category equivalence between the subvarieties of V and of W.

7) If V is generated by $\mathbf{A}$ then W is generated by $F(\mathbf{A})$.

8) If V is a congruence distributive or congruence permutable variety then W has the same properties.

Varieties can also be defined as classes of algebras of the same type which satisfy a given set of equations. If $\mathbf{A}$ is an algebra of a given type, then we may consider the variety containing of all algebras of the same type which satisfy precisely all defining equations of $\mathbf{A}$. This variety is denoted by $V(\mathbf{A})$.

A clone on a set A is a set of operations defined on A which contains all projection operations and is closed under superposition. Clones can be defined by relations, for each clone C on A there is a set R of relations on A such that $C = PolR$. For each algebra $\mathbf{A}$, we can consider the clone $T(\mathbf{A})$ which is generated by the set of all fundamental operations of $\mathbf{A}$.

Now, the concept of category equivalence can be transfered from varieties to clones. A clone C of operations defined on a set A is category equivalent to a clone $C^{'}$ of operations defined on a set B if there exist algebras $\mathbf{A}$ and $\mathbf{B}$ with universes A and B, respectively such that $C = T(\mathbf{A})$, $C^{'} = T(\mathbf{B})$ and $V(\mathbf{A})$ is category equivalent to $V(\mathbf{B})$.

Two clones C and $C^{'}$ are category equivalent if and only if the relation algebras $\underline{InvC}$ and $\underline{InvC^{'}}$ are isomorphic.

For each n-ary operation f on A, the graph of f is an $(n+1)$-ary relation $f^{\bullet} := \{(a_1, \ldots, a_n, y) \mid a_1, \ldots, a_n, y \in A \text{ and } y = f(a_1, \ldots, a_n)\}$. The concept of a graph can be naturally extended to a set of operations F by $F^{\bullet} := \{f^{\bullet} \mid f \in F\}$. A clone C is said to be a centralizer clone on A if there is a set F of operations on A such that $C = PolF^{\bullet}$.

We use the result of K. Denecke, O. Lüders [3] and R. Pöschel, L. A. Kalushnin [5] to show that a clone which is category equivalent to a centralizer clone is a centralizer clone. Moreover, we characterize centralizer clones which are category equivalent to centralizer clones of Boolean operations.

References

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