Pseudocomplements in Weak Lattices

Filip Švrček
e-mail: svrcekf@inf.upol.cz
Palacký University Olomouc, Czech Republic

1. Preliminaries

In literature, we can find several generalizations of the lattice-concept. For example, if we consider a poset such that $U(x, y) \neq \emptyset \neq L(x, y)$ for any elements $x, y \in L$ and define $x \sqcup y$, resp. $x \sqcap y$, as an arbitrary element from $U(x, y)$, resp. $L(x, y)$, with the additional condition $x \sqcup y = y$ and $x \sqcap y = x$ iff $x \leq y$, we get so-called λ-lattice (see [6]).

We can get another one from BCK-algebras, structures modeling the behavior of a connective implication derived for non-classical logics. Recall that a structure $\mathcal{A} = (A; \rightarrow, 1)$ of type $\langle 2, 0 \rangle$ is a BCK-algebra if it satifies the following axioms:
(BCK 1) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$;
(BCK 2) $x \rightarrow ((x \rightarrow y) \rightarrow y) = 1$;
(BCK 3) $x \rightarrow x = 1$;
(BCK 4) $x \rightarrow y = 1$ and $y \rightarrow x = 1$ together imply $x = y$.

If we define $x \sqcup y := (x \rightarrow y) \rightarrow y$ in $\mathcal{A}$ then this new binary operation has certain important properties that are caught in the following concept.

Definition 1. A groupoid $\mathcal{L} = (L; \sqcup)$ is called a **weak join-semilattice** if it satisfies the following:
(WJ 1) $x \sqcup x = x$;
(WJ 2) $((x \sqcup y = y)$ & $(y \sqcup x = x)) \Longrightarrow (x = y)$;
(WJ 3) $(x \sqcup y) \sqcup x = x \sqcup (x \sqcup y) = x \sqcup y$;
(WJ 4) $(x \sqcup z)) \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z$.

Proposition 2. *A groupoid $\mathcal{L} = (L; \sqcup)$ is a weak join-semilattice if and only if $\mathcal{L}$ fulfills the identity (PJ 1) and there is a partial order relation on L satisfying (PJ 2–4),*
(PJ 1) $(x \sqcup y) \sqcup y = x \sqcup y$,
(PJ 2) $x, y \leq x \sqcup y$
(PJ 3) $x \leq y \implies x \sqcup y = y$
(PJ 4) $x \leq y \implies x \sqcup z \leq y \sqcup z$.

Proof. See [4], Theorem 2.1. □

We can consider also a concept dual to that of a weak join-semillatice.

Definition 3. A groupoid $\mathcal{L} = (L; \sqcap)$ is called a **weak meet-semilattice** if the following hold in L:
(WM 1) $x \sqcap x = x$;
(WM 2) $((x \sqcap y = x)$ & $(y \sqcap x = y)) \Longrightarrow (x = y)$;
(WM 3) $(x \sqcap y) \sqcap x = x \sqcap (x \sqcap y) = x \sqcap y$;
(WM 4) $(z \sqcap (x \sqcap y)) \sqcap (z \sqcap y) = z \sqcap (x \sqcap y)$.

Proposition 4. *A groupoid* $\mathcal{L} = (L; \sqcap)$ *is a weak meet-semilattice if and only if* $\mathcal{L}$ *fulfills the identity (PM 1) and there is a partial order relation on* L *satisfying (PM 2–4),*

(PM 1) $x \sqcap (x \sqcap y) = x \sqcap y,$
(PM 2) $x \sqcap y = x, y$
(PM 3) $x \leq y \implies x \sqcap y = x$
(PM 4) $x \leq y \implies z \sqcap x \leq z \sqcap y.$

Proof. See [4], Corollary 3.1. □

Definition 5. An algebra $\mathcal{L} = (L; \sqcup, \sqcap)$ of the type $\langle 2, 2\rangle$ is called a **weak lattice** if it is both a weak meet- and join-semilattice and if it satisfies the absorption laws

(W1) $x \sqcap (x \sqcup y) = x,$
(W2) $(x \sqcap y) \sqcup y = y.$

Proposition 6. *An algebra* $\mathcal{L} = (L; \sqcup, \sqcap)$ *of the type* $\langle 2, 2\rangle$ *is a weak lattice if and only if* $\mathcal{L}$ *fulfills identities (PJ 1) and (PM 1) and if its induced order satisfies (PJ 2–4) and (PM 2–4).*

Proof. See [4], Lemma 4.1. □

2. Pseudocomplemented weak lattices

Definition 7. Let $\mathcal{L} = (L; \sqcap, 0)$ be a weak meet-semilattice with the least element 0. If there is a greatest element $x \in L$ such that

$$x \sqcap a = 0, \quad \text{or} \quad a \sqcap x = 0 \quad \text{respectively,}$$

then x is called a **left**, or a **right pseudocomplement** of the element $a \in L$ respectively. We will denote it a^*_L, respectively a^*_R.

If a^*_L, respectively a^*_R exists for each $x \in L$, we say that $\mathcal{L}$ is **left-**, respectively **right-pseudocomplemented**.

Remark 8. For the sake of simplicity, we will write x^{**}_{RL} for $(x^*_R)^*_L$.

Lemma 9. *Let* $\mathcal{L} = (L; \sqcap, 0)$ *be both a left- and right-pseudocomplemented weak meet-semilattice. Then for arbitrary* $a, b \in L$

(i) $a \leq a^{**}_{RL}, a^{**}_{LR},$
(ii) $a \leq b \implies b^*_L \leq a^*_L,$
(iii) $a^*_L = a^{***}_{LRL},$
(iv) $0^*_R = 0^*_L$ *is the greatest element 1 in* $\mathcal{L}$,
(v) $1^*_L = 0,$
(vi) $a = b^*_L \implies a = a^{**}_{RL}.$

Definition 10. We say that a weak meet-semilattice $\mathcal{L} = (L; \sqcap, 0)$ with the least element 0 is **pseudocomplemented** if it is both left- and right-pseudocomplemented and if $x^*_L = x^*_R$ holds for each element $x \in L$.

Remark 11. Let us denote $L^* = \{x^*;\ x \in L\}$ in any pseudocomplemented weak meet-semilattice $\mathcal{L} = (L; \sqcap, {}^*, 0)$. We can show that L^* need not be a subalgebra of $\mathcal{L}$.

Proposition 12. *Let* $\mathcal{L} = (L; \sqcap, {}^*, 0)$ *be a pseudocomplemented weak meet-semilattice. If we define binary operations* $\cup, \cap$ *on* L^* *by*

$$x \cup y := (y^* \sqcap x^*)^*,$$
$$x \cap y := (x \sqcap y)^{**},$$

then the algebra $(L^*; \cup, \cap)$ *is a weak lattice.*

Theorem 13. *For a given pseudocomplemented weak meet-semilattice* $\mathcal{L} = (L; \sqcap, {}^*, 0)$ *the structure* $(L^*; \cup, \cap, {}^*, 0, 1)$ *is a Boolean algebra.*

Proposition 14. *Let* $\mathcal{L} = (L; \sqcap, {}^*, 0)$ *be a pseudocomplemented weak meet-semilattice. Let us define a binary relation* Θ *on* L *as follows*

$$(x, y) \in \Theta \iff x^* = y^*. \tag{$*$}$$

Then Θ *is an equivalence on* L *and for any* $a \in L$ *the following holds:*
(a) The element a^{**} *is the greatest one in the class* $[a]_\Theta$*.*
(b) The class $[a]_\Theta$ *contains just one element from* L^* *which is* a^{**}*.*

In any pseudocomplemented weak meet-semilattice $\mathcal{L} = (L; \sqcap, {}^*, 0)$, we can consider the set of so-called **dense elements** $D(L)$, i.e. the elements whose pseudocomplements are equal to zero, i.e. $D(L) = \{x \in L; x^* = 0\}$.

Theorem 15. *Let* $\mathcal{L} = (L; \sqcap, {}^*, 0)$ *be a pseudocomplemented weak meet-semilattice and* Θ *the equivalence on* L *given by* ($*$)*. Define operations* $\Cup, \Cap, {}^\star$ *on* L/Θ *as follows*

$$[x]_\Theta \Cup [y]_\Theta = [(y^* \sqcap x^*)^*]_\Theta,$$
$$[x]_\Theta \Cap [y]_\Theta = [x^{**} \sqcap y^{**}]_\Theta,$$
$$[x]_\Theta^\star = [x^*]_\Theta$$

Then $\mathcal{L}/\Theta = (L/\Theta; \Cup, \Cap, {}^\star, \{0\}, D(L))$ *is a Boolean algebra isomorphic to* $(L^*; \cup, \cap, {}^*, 0, 1)$*.*

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