C-clones and C-monoids.

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In this paper a restricted version of the Galois connection between polymorphisms and invariants, called $Pol - C\text{Inv}$, is studied, where the invariant relations are restricted to so called clausal relations. The lattice of all clones arising from this Galo is connection, so called C-clones, is investigated up to equality of their unary parts, which are denominated C-monoids.

Introduction

In this paper we study a special set $C\mathrm{R}_D$ of relations on a finite set D, called *clausal relations* (see Definition 4). The latter were already investigated by E. Vargas (see [3]). The definition of clausal relations is base d on the notion of a *clausal constraint* as a disjunction of inequalities of the form $x \geq d$ and $x \leq d$, where $d \in D = \{0, 1, \ldots, n-1\}$ are constants and x belongs to a set X of variables. These were studied by N. Creignou, M. Hermann, A . Krokhin and G. Salzer (see [1]).

A *clone* on a set D is a set of finitary operations on D that is closed under composition and contains all projections. It is well known (see [2]) that on a finite set D the Galois closed classes of operations w.r.t. the Galois connec tion of polymorphisms and invariant relations are exactly all clones on D. In other words, every clone F on D can be described as $F = PolQ$ for some set Q of finitary relations.

In this paper we are interested in describing clones that are determined by sets of clausal relations, i.e. in so called *C-clones* (see Definition 10). For $|D| \geq 3$ in [3] the existence of an infinite number of C-clones has been proven, thus to describe all of them seems to be rather difficult. Therefore, we now turn our attention towards monoids and weak Krasner clones that are determined by sets of clausal relations, i.e. *C-monoids* (see Definition 12). In particular, this is a contribution to the structure of the lattice of all clones.

Throughout the paper, $\mathbb{N} = \{0, 1, 2, \ldots\}$ will denote the set of natural numbers, and $\mathbb{N}_+ = \{1, 2, \ldots\}$ the set of positive natural numbers. Furthermore, $D = \{0, 1, \ldots, n-1\}$ for a fixed natural number $n \geq 2$.

1. C-clones

In this section we provide some definitions, notations and conventions used hereafter.

Definition 1. Let $m \in \mathbb{N}_+$. An *m-ary relation* ϱ on D is a subset of the m-fold Cartesian product D^m. By $\mathrm{R}_D^{(m)} := \mathcal{P}(D^m)$ we denote the *set of all m-ary relations* defined on D and by $\mathrm{R}_D := \bigcup_{m=1}^{\infty} \mathrm{R}_D^{(m)}$ the *set of all finitary relations* on D.

Definition 2. Let $m \in \mathbb{N}_+$, and $\theta \in \mathrm{Eq}(m)$ be an equivalence relation on the set $m = \{1, \ldots, m\}$. We define $d_\theta \in \mathrm{R}_D^{(m)}$ to be the relation

$$d_\theta := \{(x_1, \ldots, x_m) \in D^m \mid \forall (i, j) \in \vartheta : \ x_i = x_j\}$$

and call it a *trivial* or *diagonal relation*. The set of all diagonal relations together with the empty relation $\emptyset$ is denoted by $\mathrm{diag}(D)$.

Definition 3. [3] Let $p, q \in \mathbb{N}_+$. For given $\mathbf{a} = (a_1, \ldots, a_p) \in D^p$ and $\mathbf{b} = (b_1, \ldots, b_q) \in D^q$, the *clausal relation* $\mathrm{R}_{\mathbf{b}}^{\mathbf{a}}$ *of type* (p, q), is the set of all tuples $(x_1, \ldots, x_p, y_1, \ldots, y_q) \in D^{p+q}$ satisfying

$$(x_1 \geq a_1) \vee \cdots \vee (x_p \geq a_p) \vee (y_1 \leq b_1) \vee \cdots \vee (y_q \leq b_q). \tag{1}$$

Observe that if $a_i = 0$ for some $i \in \{1, \ldots, p\}$ or $b_j = n - 1$ for some $j \in \{1, \ldots, q\}$, then the relation $\mathrm{R}_{\mathbf{b}}^{\mathbf{a}}$ is the full Cartesian power of D, i.e. $\mathrm{R}_{\mathbf{b}}^{\mathbf{a}} = D^{p+q}$, because (1) is satisfied for any $(x_1, \ldots, x_p, y_1, \ldots, y_q) \in D^{p+q}$.

Definition 4. Let $p, q \in \mathbb{N}_+$. We use

$$\mathcal{R}_q^p := \{\mathrm{R}_{\mathbf{b}}^{\mathbf{a}} \mid \mathbf{a} \in D^p, \mathbf{b} \in D^q\}$$

to denote the *set of all clausal relations of arity*[1] $p + q$ and

$$C\mathrm{R}_D := \bigcup_{(p,q) \in \mathbb{N}_+^2} \mathcal{R}_q^p$$

for the set of all *finitary clausal relations* on D.

The following lemma shows that the only trivial clausal relations are those we already noticed after Definition 3, and that the non-trivial ones can be easily identified by their parameters **a** and **b**.

Lemma 5. [3] *The set* $C\mathrm{R}_D$ *can be partitioned as*

$$C\mathrm{R}_D = \{D^{(p+q)} \mid p, q \in \mathbb{N}_+\} \,\dot{\cup}\, C\mathrm{R}_D^*,$$

where

$$\{D^{(p+q)} \mid p, q \in \mathbb{N}_+\} = C\mathrm{R}_D \cap \mathrm{diag}(D)$$

are the trivial clausal relations and

$$C\mathrm{R}_D^* = \{\mathrm{R}_{\mathbf{b}}^{\mathbf{a}} \mid \mathbf{a} \in (D \setminus \{0\})^p, \mathbf{b} \in (D \setminus \{n-1\})^q; p, q \in \mathbb{N}_+\}$$

are the non-trivial clausal relations.

Definition 6. Let $n \in \mathbb{N}_+$. An *n-ary operation* on the domain D is a function $f : D^n \longrightarrow D$. We denote by $\mathrm{O}_D^{(n)} := \{f \mid f : D^n \longrightarrow D\}$ the *set of all n-ary operations on D* and by $\mathrm{O}_D := \bigcup_{n=1}^{\infty} \mathrm{O}_D^{(n)}$ the *set of all finitary operations on D*.

[1] If we speak of a clausal relation of arity $p + q$, we implicitly mean also that the clausal relation is of type (p, q).

Definition 7. Let $m, n \in \mathbb{N}_+$. We say that an n-ary operation $f \in \mathrm{O}_D^{(n)}$ *preserves* an m-ary relation $\varrho \in \mathrm{R}_D^{(m)}$, denoted by $f \rhd \varrho$, if whenever

$$r_1 = (a_{11}, \ldots, a_{m1}) \in \varrho, \ldots, r_n = (a_{1n}, \ldots, a_{mn}) \in \varrho$$

it follows that also f applied to these tuples belongs to ϱ, i.e.

$$f \circ (r_1, \ldots, r_n) := (f(a_{11}, \ldots, a_{1n}), \ldots, f(a_{m1}, \ldots, a_{mn})) \in \varrho.$$

This binary relation $\rhd$ in a natural way induces a Galois connection between finitary operations and finitary relations on D.

Definition 8. Let $F \subseteq \mathrm{O}_D$ be a set of operations on D. Then we define $\mathrm{Inv}_D F$ as the set of all relations that are *invariant* for all $f \in F$:

$$\mathrm{Inv}_D F := \{\varrho \in \mathrm{R}_D \mid \forall f \in F : f \rhd \varrho\}.$$

Similarly, for a set $Q \subseteq \mathrm{R}_D$ of relations, we define

$$Pol_D Q := \{f \in F \mid \forall \varrho \in Q : f \rhd \varrho\}$$

as the *set of polymorphisms of* Q. Furthermore, for $n \in \mathbb{N}_+$ we abbreviate

$$Pol_D^{(n)} Q := \mathrm{O}_D^{(n)} \cap Pol_D Q.$$

If D is known from the context we write Pol instead of Pol_D, and Inv instead of Inv_D. The operators Pol and Inv define the Galois connection $Pol - \mathrm{Inv}$. We can restrict this Galois connection on the operational side,

$$End Q = \mathrm{O}_D^{(1)} \cap Pol Q$$

and call these functions *endomorphisms of* Q.

Next we present a restriction of the Galois connection $Pol - \mathrm{Inv}$ where the relations are confined to be clausal relations. This restriction yields a much smaller number of Galois closed sets of operations, so called C-clones.

Definition 9. For $F \subseteq \mathrm{O}_D$ we define $C\mathrm{Inv}\, F := \mathrm{Inv}\, F \cap C\mathrm{R}_D$. The operators

$$C\mathrm{Inv} : \mathcal{P}(\mathrm{O}_D) \longrightarrow \mathcal{P}(C\mathrm{R}_D) : \qquad F \mapsto C\mathrm{Inv}\, F$$

and

$$Pol : \mathcal{P}(C\mathrm{R}_D) \longrightarrow \mathcal{P}(\mathrm{O}_D) : \qquad Q \mapsto Pol Q$$

define a *Galois connection* $Pol - C\mathrm{Inv}$ between operations and clausal relations.

Definition 10. A set $F \subseteq \mathrm{O}_D$ of operations is called a C*-clone* if $F = Pol Q$ for some set $Q \subseteq C\mathrm{R}_D$ of clausal relations.

2. C-MONOIDS

As we already mentioned our aim is the analysis of the lattice of all C-clones. This seems to be fairly hard as [3] exhibited already infinitely many C-clones for $|D| \geq 3$. A step towards our goal is to describe some C-clones correlating with weak Krasner clones on the relational side.

Definition 11. For $M \subseteq \mathrm{O}_D^{(1)}$ we define $C\mathrm{Inv}\, M := \mathrm{Inv}\, M \cap C\mathrm{R}_D$. The operators

$$C\mathrm{Inv} : \mathcal{P}(\mathrm{O}_D^{(1)}) \longrightarrow \mathcal{P}(C\mathrm{R}_D) : \qquad M \mapsto C\mathrm{Inv}\, M$$
$$End : \mathcal{P}(C\mathrm{R}_D) \longrightarrow \mathcal{P}(\mathrm{O}_D^{(1)}) : \qquad Q \mapsto EndQ$$

define a *Galois connection* $End - C\mathrm{Inv}$ between endomorphisms and clausal relations.

The Galois closed sets of operations of this Galois connection are defined as follows:

Definition 12. A set $M \subseteq \mathrm{O}_D^{(1)}$ is called a *clausal monoid* for short *C-monoid* if $M = EndQ$ for some set $Q \subseteq C\mathrm{R}_D$ of clausal relations.

References

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[3] E. Vargas.: *Clausal relations and C-clones.* J. Discuss. Math. Gen. Algebra Appl., (2010), to appear.