Some semilattice decompositions of dimonoids

Anatolii Zhuchok
e-mail: zhuchok_a@mail.ru
Kyiv National Taras Shevchenko University, Ukraine

The notion of a non-commutative Lie algebra (Leibniz algebra) appeared in the researches on a homology theory for Lie algebras [1]. It is well-known that for Lie algebras there is a notion of an universal enveloping associative algebra. Jean-Louis Loday [2] found an universal enveloping algebra for Leibniz algebras. A role of such object plays dialgebras, that is vector spaces with two bilinear associative operations $\prec$ and $\succ$ satisfying the following axioms:

$$(x \prec y) \prec z = x \prec (y \succ z), \qquad (1)$$

$$(x \succ y) \prec z = x \succ (y \prec z), \qquad (2)$$

$$(x \prec y) \succ z = x \succ (y \succ z) \qquad (3)$$

for all $x, y, z \in D$. Dialgebras investigated in the different papers. So recently Kolesnikov [3] shown that any dialgebra can be obtained from some associative conformal algebra. The conformal algebras were introduced by Kac [4] as a formal language of th e description of properties of algebraic structures occurring in mathematical physics. The connection between the dialgebras and conformal algebras allowed to introduce the notion of a variety of dialgebras [3] with the help of the notion of a operad [5].

A set D equipped with two binary associative operations $\prec$ and $\succ$ satisfying the axioms (1)-(3) is called a dimonoid [2]. So dialgebra is a linear analogue of a dimonoid. In our time dimonoids have became a standart tool in the theory of Leibniz algebras. One of the first results about dimonoids is the description of a free dimonoid generated by a given set [2]. With the help of the properties of free dimonoids it was described free dialgebras and investigated a cohomology of dialgebras [2]. In [6] Liu used the notion of a dimonoid to introduce the notion of an one-sided diring and studied basic properties of dirings. The noti on of a diband of subdimonoids was introduced in [7]. This notion generalizes the notion of a band of semigroups [8] and is effective to describe structural properties of dimonoids. In terms of dibands of subdimonoids, in particular, it was proved that eve ry commutative dimonoid is a semilattice of archimedean subdimonoids [7]. In [9] it was formulated that every idempotent dimonoid is a semilattice of rectangular subdimonoids. Some new dialgebras were introduced in terms of dimonoids in [2]. It is also well-known that the notion of a dimonoid generalizes the notion of a digroup [10]. Pirashvili [11] considered duplexes which are sets with two binary as sociative operations and described a free duplex. Dimonoids in the sense of Loday [2] are examples of duplexes.

Obviously if the operations of a dimonoid coincide then it becomes a semigroup. Therefore studying of dimonoids via semigroup techniques may constitute a research direction.

The purpose of this work is to obtain some semilattice decompositions of dimonoids with a commutative operation.

Examples of dimonoids were given in [2, 7].

A dimonoid $(D, \prec, \succ)$ will be called an idempotent dimonoid or a diband if $x \prec x = x = x \succ x$ for all $x \in D$. Let J be some idempotent dimonoid. We call a dimonoid $(D, \prec, \succ)$ a diband J of subdimonoids D_i $(i \in J)$ if $D = \bigcup_{i \in J} D_i$, $D_i \bigcap D_j = \varnothing$ for $i \neq j$ and $D_i \prec D_j \subseteq D_{i \prec j}$, $D_i \succ D_j \subseteq D_{i \succ j}$ for all $i, j \in J$. If J is a band (= idempotent semigroup), then we say that $(D, \prec, \succ)$ is a band J of subdimonoids D_i $(i \in J)$. If J is a commutative band, then we say that $(D, \prec, \succ)$ is a semilattice J of subdimonoids D_i $(i \in J)$.

A commutative idempotent semigroup is called a semilattice.

Lemma 1. The operations of a dimonoid $(D, \prec, \succ)$ coincide if one of the following conditions holds:

(i) $(D, \prec)$ is a semilattice;

(ii) $(D, \prec)$ is a left cancellative (cancellative) semigroup;

(iii) $(D, \prec)$ is a commutative separarive semigroup;

(iv) $(D, \prec)$ is a commutative globally idempotent semigroup.

As usual N denotes the set of positive integers. Let $(D, \prec, \succ)$ be a dimonoid and $a \in D$, $n \in N$. Denote by a^n (respectively, $n\,a$) the degree n of an element a concerning the operation $\prec$ (respectively, $\succ$).

Lemma 2. Let $(D, \prec, \succ)$ be an arbitrary dimonoid. For all $x, y, t \in D, n \in N$

(i) $(x \prec y)^n \succ t = n(x \succ y) \succ t = n(x \prec y) \succ t$;

(ii) $t \prec n(x \succ y) = t \prec (x \prec y)^n = t \prec (x \succ y)^n$.

Let S be a semigroup and $a \in S$. The elements $x, y \in S$ are called a-connected if there exist $n, m \in N$ such that $(xa)^n \in yaS$ and $(ya)^m \in xaS$. The semigroup S is a-connected if x, y are a-connected for all $x, y \in S$. Recall t hat a semigroup S is called archimedean if for any $a, b \in S$ there exists $n \in N$ such that b^n belongs to the principial two-sided ideal $J(a)$ generated by a.

Lemma 3. Let $(D, \prec, \succ)$ be a dimonoid and $a \in D$ be an arbitrary fixed element. Then

(i) If $(D, \prec)$ is a a-connected semigroup, then $(D, \succ)$ is a a-connected semigroup;

(ii) $(D, \prec)$ is an archimedean semigroup if and only if $(D, \succ)$ is an archimedean semigroup.

We say that a dimonoid is archimedean if its both semigroups are archimedean.

Theorem 4. Every dimonoid $(D, \prec, \succ)$ with a commutative operation $\prec$ is a semilattice Y of archimedean subdimonoids $D_i,\ i \in Y$.

This theorem extends Theorem 2 from [7] about the decomposition of commutative dimonoids (=dimonoids with commutative operations) into semilattices of archimedean subdimonoids.

Let $(D, \prec, \succ)$ be a dimonoid and $a \in D$. A dimonoid $(D, \prec, \succ)$ will be called a-connected if semigroups $(D, \prec)$ and $(D, \succ)$ are a-connected.

Theorem 5. Let $(D, \prec, \succ)$ be a dimonoid with a commutative operation $\prec$ and let $a \in D$ be an arbitrary fixed element. Then $(D, \prec, \succ)$ is a semilattice T_a of a-connected subdimonoids $D_i,\ i \in T_a$.

References

[1] Loday J.-L. : *Algèbres ayant deux opérations associatives (digèbres)*, C. R. Acad. Sci. Paris 321 (1995), 141-146.

[2] Loday J.-L. : *Dialgebras*, In: Dialgebras and related operads, Lecture Notes in Math. 1763, Springer, Berlin, 2001, 7-66.
[3] Kolesnikov P.S. : *Varieties of dialgebras and conformal algebras*, Sib. Math. Journal, 2008. V. 49, 2., 322-339 (In Russian).
[4] Kac V.G. : *Vertex Algebras for Beginners*, University Lecture Series, vol. 10, AMS, Providence, RI, 1996.
[5] Ginzburg V., Kapranov M. : *Kozul duality for operads*, Duke Math. J. 76 (1994) no. 1, 203-272.
[6] Liu K. : *A Class of Ring-like Objects*, preprint, available at http://arxiv.org/abs/math/0411586v2.
[7] Zhuchok A.V. : *Commutative dimonoids*, Algebra and Discrete Mathematics, N.2 (2009), 116-127.
[8] Clifford A.H. : *Bands of semigroups*, Proc. Amer. Math. Soc., 5 (1954), 499–504.
[9] Zhuchok A.V. : *On idempotent dimonoids*, International Conference on Semigroups and related topics, Porto, Portugal, 2009, p. 87.
[10] Phillips J. D. : *A short basis for the variety of digroups*, Semigroup Forum, 70 (2005), 466-470.
[11] Pirashvili T. : *Sets with two associative operations*, Central European Journal of Mathematics, 2 (2003), 169-183.