Term functions of direct products

Erhard Aichinger and Peter Mayr

Department of Algebra
Johannes Kepler University Linz, Austria

June 2014, AAA88

Supported by the Austrian Science Fund (FWF)
P24077 and P24285

Main question in this talk

How do the term functions of $\mathbf{A} \times \mathbf{B}$ depend on the term functions of $\mathbf{A}$ and $\mathbf{B}$?

$$\begin{aligned} \mathrm{Clo}(\mathbf{A}) &= \text{set of term functions,} \\ \mathrm{Pol}(\mathbf{A}) &= \text{set of polynomial functions.} \end{aligned}$$

The desired theorem

Let $\mathbf{A}, \mathbf{B}$ be similar algebras. We assume $[\ldots]$. Then $\mathrm{Clo}_k(\mathbf{A} \times \mathbf{B}) = \mathrm{Clo}_k\mathbf{A} \times \mathrm{Clo}_k\mathbf{B}$.

Defining independence of **A** and **B**

What does $\mathrm{Clo}_k(\mathbf{A} \times \mathbf{B}) = \mathrm{Clo}_k\mathbf{A} \times \mathrm{Clo}_k\mathbf{B}$ mean?

Proposition

Let **A**, **B** be similar finite algebras, $k \in \mathbb{N}$. TFAE:

1. $\Phi : \mathrm{Clo}_k(\mathbf{A} \times \mathbf{B}) \to \mathrm{Clo}_k(\mathbf{A}) \times \mathrm{Clo}_k(\mathbf{B})$, $\Phi(t^{\mathbf{A}\times\mathbf{B}}) = (t^{\mathbf{A}}, t^{\mathbf{B}})$ is surjective.
2. For all k-variable terms s and t, there is a term u such that

$$\begin{aligned} u^{\mathbf{A}} &= s^{\mathbf{A}} \\ u^{\mathbf{B}} &= t^{\mathbf{B}}. \end{aligned}$$

Independent groups

Proposition

Let $\mathbf{G}, \mathbf{H}$ be finite groups of coprime order. Then for all terms s, t there is a term u with

$$u^{\mathbf{G}} = s^{\mathbf{G}} \text{ and } u^{\mathbf{H}} = t^{\mathbf{H}}.$$

Proof by example:

- Assume $\exp \mathbf{G} = 18, \quad \exp \mathbf{H} = 5$.
- Let
$$s := xyxx \text{ and } t := yxy.$$
- Consider
$$u := x^{55} yxy^{36} x^{55}.$$
- Then
$$u \equiv_{\mathbf{G}} xyxx \text{ and } u \equiv_{\mathbf{H}} yxy.$$

Necessary conditions for independence

Definition

A, **B** similar algebras. Then **A** and **B** are independent if for all 2-variable terms s and t, there is u with $u^{\mathbf{A}} = s^{\mathbf{A}}$ and $u^{\mathbf{B}} = t^{\mathbf{B}}$.

Lemma

A, **B** similar independent algebras. Then for every subalgebra $\mathbf{E} \leq \mathbf{A} \times \mathbf{B}$, we have

$$\mathbf{E} = \pi_{\mathbf{A}}(\mathbf{E}) \times \pi_{\mathbf{B}}(\mathbf{E}).$$

Every subalgebra of **E** is a product subalgebra.

Necessary conditions for independence

Lemma

Let **A**, **B** be similar independent algebras, $\mathbf{E} \leq \mathbf{A}$, $\mathbf{F} \leq \mathbf{B}$. Then for every $\alpha \in \text{Con}(\mathbf{E} \times \mathbf{F})$, there are $\beta \in \text{Con}(\mathbf{E})$, $\gamma \in \text{Con}(\mathbf{F})$ such that for all $a, c \in E$, $b, d \in F$:

$$\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right) \in \alpha \text{ iff } (a, c) \in \beta \text{ and } (b, d) \in \gamma.$$

Every congruence of $\mathbf{E} \times \mathbf{F}$ is a product congruence.

Theorem (EA, Mayr, Opršal, 2014)

Let $\mathbf{A}, \mathbf{B}$ be similar Mal'cev algebras, and let $m, n \in \mathbb{N}_0$. Assume that

1. all subalgebras of $\mathbf{A} \times \mathbf{B}$ are product subalgebras,
2. for all subalgebras $\mathbf{E}$ of $\mathbf{A}$ and $\mathbf{F}$ of $\mathbf{B}$, all congruences of $\mathbf{E} \times \mathbf{F}$ are product congruences.

Then all subalgebras of $\mathbf{A}^m \times \mathbf{B}^n$ are product subalgebras.

Then $\forall \mathbf{C} \leq \mathbf{A}^m \times \mathbf{B}^n \quad \exists \mathbf{E} \leq \mathbf{A}^m, \ \mathbf{F} \leq \mathbf{B}^n \quad : \quad \mathbf{C} = \mathbf{E} \times \mathbf{F}$.

Proof:

We have to show

$$\forall \mathbf{C} \leq \mathbf{A}^m \times \mathbf{B}^n \quad \exists \mathbf{E} \leq \mathbf{A}^m, \ \mathbf{F} \leq \mathbf{B}^n \quad : \quad \mathbf{C} = \mathbf{E} \times \mathbf{F}.$$

- Let $\mathbf{C} \leq \mathbf{A}^m \times \mathbf{B}^n$.
- We show $\mathbf{C} = \pi_{A^m}(\mathbf{C}) \times \pi_{B^n}(\mathbf{C})$.
- We proceed by induction on $m + n$.
- The case $m = 0$ or $n = 0$ is easy.
- The case $m = n = 1$ follows from the assumptions.

Proof (induction step):

We show $\mathbf{C} = \pi_{A^m}(\mathbf{C}) \times \pi_{B^n}(\mathbf{C})$.

- Let $(\mathbf{a}, \mathbf{b}) \in \pi_{A^m}(\mathbf{C}) \times \pi_{B^n}(\mathbf{C})$.
- By the induction hypothesis, $\exists c \in A, d \in B$ s.t.

$$\begin{array}{llll} ((a_1, \dots, a_{m-1}, & c), & (b_1, \dots, b_{n-1}, & b_n)) & \in & C \\ ((a_1, \dots, a_{m-1}, & a_m), & (b_1, \dots, b_{n-1}, & d)) & \in & C \end{array}$$

- Define $\alpha \subseteq (A \times B)^2$ (a set of forks) by

$$\begin{aligned} \alpha := \{((x_m, y_n), (x'_m, y'_n)) \mid \\ ((x_1, \dots, x_{m-1}, x_m), (y_1, \dots, y_{n-1}, y_n)) \in C \text{ and} \\ ((x_1, \dots, x_{m-1}, x'_m), (y_1, \dots, y_{n-1}, y'_n)) \in C\}. \end{aligned}$$

- α is a congruence of a subalgebra $\mathbf{S} \leq \mathbf{A} \times \mathbf{B}$.
- $((c, b_n), (a_m, d)) \in \alpha$. Hence $((c, d), (a_m, d)) \in \alpha$.

Proof (induction step):

- From $((c,d),(a_m,d)) \in \alpha$, we obtain $\mathbf{u} \in A^{m-1}$, $\mathbf{v} \in B^{n-1}$ s.t.
$$\begin{array}{lll} ((\mathbf{u},c), & (\mathbf{v},b_n)) & \in & C \\ ((\mathbf{u},c), & (\mathbf{v},d)) & \in & C. \end{array}$$
- We already had (induction hypothesis)
$$((\mathbf{a},a_m), \quad (\mathbf{b},d)) \quad \in \quad C.$$
- Mal'cev yields
$$((a_1,\dots,a_m),(b_1,\dots,b_n)) \in C.$$

Application to term functions

Corollary (EA, Mayr, 2014)

Let **A**, **B** be similar finite Mal'cev algebras, $k \in \mathbb{N}$. We assume:

1. All subalgebras of $\mathbf{A} \times \mathbf{B}$ are product subalgebras.
2. For all subalgebras **E** of **A** and **F** of **B**, all congruences of $\mathbf{E} \times \mathbf{F}$ are product congruences.

Then $\Phi : \mathrm{Clo}_k(\mathbf{A} \times \mathbf{B}) \to \mathrm{Clo}_k(\mathbf{A}) \times \mathrm{Clo}_k(\mathbf{B})$,

$$\Phi(t^{\mathbf{A}\times\mathbf{B}}) = (t^{\mathbf{A}}, t^{\mathbf{B}}) \text{ for all terms } t$$

is a bijection.

Proof:

Consider $\mathbf{D} \leq \mathbf{A}^{A^k} \times \mathbf{B}^{B^k}$ with $D := \{(u^{\mathbf{A}}, u^{\mathbf{B}}) \mid u \text{ is a term }\}$.
Then $\mathbf{D} = \pi_A(\mathbf{D}) \times \pi_B(\mathbf{D}) = \mathrm{Clo}_k(\mathbf{A}) \times \mathrm{Clo}_k(\mathbf{B})$.

Application to polynomial functions

Corollary

Let $\mathbf{A}, \mathbf{B}$ be similar finite Mal'cev algebras, $k \in \mathbb{N}$. We assume:

All congruences of $\mathbf{A} \times \mathbf{B}$ *are product congruences.*

Then $\Phi : \mathrm{Pol}_k(\mathbf{A} \times \mathbf{B}) \to \mathrm{Pol}_k(\mathbf{A}) \times \mathrm{Pol}_k(\mathbf{B})$, $\Phi(p) = ([p]_{\nu_1}, [p]_{\nu_2})$ is a bijection.

Proof:

For every $a \in A$, $b \in B$, add a constant operation $c_{(a,b)}$ with $c_{(a,b)}^{\mathbf{A} \times \mathbf{B}} = (a, b)$. Then apply the previous theorem.

The corollary was conjectured in [Pilz, 1980]. It was proved in [Aichinger, 2001] for finite expanded groups, and in [Kaarli and Mayr, 2010] for finite Mal'cev algebras and for finite algebras with majority term. It does not generalize to CD varieties.

Generalisation

Edge terms

For $k \geq 3$, a $(k+1)$-ary term is a *k-edge term* on **A** if for all $a, b \in A$:

$$\begin{aligned} t^{\mathbf{A}}(a, a, b, b, b, \ldots, b) &= b \\ t^{\mathbf{A}}(a, b, a, b, b, \ldots, b) &= b \\ t^{\mathbf{A}}(b, b, b, a, b, \ldots, b) &= b \\ &\ddots \\ t^{\mathbf{A}}(b, b, b, b, b, \ldots, a) &= b \end{aligned}$$

Theorem [Berman et al., 2010]

A finite algebra. **A** has an edge term $\Leftrightarrow \exists$ polynomial p $\forall n \in \mathbb{N}$: $|\mathrm{Sub}(\mathbf{A}^n)| \leq 2^{p(n)}$. (**A** has few subpowers).

Independent algebras with edge term

Theorem (EA, Mayr, 2014)

Let $\mathbf{A}, \mathbf{B}$ be algebras in a variety with k-edge term. Assume:

1. For all $r, s \in \mathbb{N}$ with $r + s \leq \max(2, k-1)$, all subalgebras of $\mathbf{A}^r \times \mathbf{B}^s$ are product subalgebras.
2. For all $\mathbf{E} \leq \mathbf{A}, \mathbf{F} \leq \mathbf{B}$, all tolerances of $\mathbf{E} \times \mathbf{F}$ are product tolerances.

Then for all $m, n \in \mathbb{N}$, all subalgebras of $\mathbf{A}^m \times \mathbf{B}^n$ are product subalgebras.

Application to term functions

Corollary

Let $\mathbf{A}, \mathbf{B}$ be finite algebras in a variety V with k-edge term. Assume

1. for all $r, s \in \mathbb{N}$ with $r + s \leq \max(2, k-1)$, all subalgebras of $\mathbf{A}^r \times \mathbf{B}^s$ are product subalgebras
2. for all $E \leq \mathbf{A}, F \leq \mathbf{B}$, all tolerances of $\mathbf{E} \times \mathbf{F}$ are product tolerances.

Let $n \in \mathbb{N}$, and let s, t be n-variable terms. Then there is a term u with $\mathrm{u}^{\mathbf{A}} = \mathrm{s}^{\mathbf{A}}$ and $\mathrm{u}^{\mathbf{B}} = \mathrm{t}^{\mathbf{B}}$.

Aichinger, E. (2001).
On near-ring idempotents and polynomials on direct products of Ω-groups.
Proc. Edinburgh Math. Soc. (2), 44:379–388.

Berman, J., Idziak, P., Marković, P., McKenzie, R., Valeriote, M., and Willard, R. (2010).
Varieties with few subalgebras of powers.
Transactions of the American Mathematical Society, 362(3):1445–1473.

Kaarli, K. and Mayr, P. (2010).
Polynomial functions on subdirect products.
Monatsh. Math., 159(4):341–359.

Pilz, G. F. (1980).
Near-rings of compatible functions.
Proceedings of the Edinburgh Mathematical Society, 23:87–95.