TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

On the relationship of maximal clones and maximal C-clones

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Outline

Presenting joint work with. . .

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School of Mathematics.

Outline

1. Galois theory for clones
2. Clausal relations and clausal clones
3. Maximal clones/C-clones

Galois theory for clones

Basis

Galois correspondence connecting
finitary operations on $D \longleftrightarrow$ finitary relations on D

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Finitary operations

- For $k \in \mathbb{N}_+$ any $f\colon D^k \longrightarrow D$ is a k-ary operation on D
- $\mathrm{O}_D^{(k)} := D^{D^k}$ set of k-ary operations on D
- $\mathrm{O}_D := \bigcup_{k\in\mathbb{N}_+} \mathrm{O}_D^{(k)}$ set of all finitary operations on D

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Finitary relations

- For $m \in \mathbb{N}_+$ subsets $\varrho \subseteq D^m$ are m-ary relations on D
- $\mathrm{R}_D^{(m)} := \mathfrak{P}(D^m)$ set of m-ary relations on D
- $\mathrm{R}_D := \bigcup_{m\in\mathbb{N}_+} \mathrm{R}_D^{(m)}$ set of all finitary relations on D

Preservation condition

$m, n \in \mathbb{N}_+,\ f \in \mathrm{O}_D^{(n)},\ \varrho \in \mathrm{R}_D^{(m)}$

$$\begin{matrix} x_{0,0} & \cdots & x_{0,n-1} \\ \vdots & \ddots & \vdots \\ x_{m-1,0} & \cdots & x_{m-1,n-1} \end{matrix}$$

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Truth of this condition: $f \rhd \varrho$

Polymorphisms and invariant relations

For $F \subseteq \mathrm{O}_D$ and $Q \subseteq \mathrm{R}_D$:

$$\mathrm{Inv}_D\, F := \{\varrho \in \mathrm{R}_D \mid \forall f \in F : \quad f \rhd \varrho\}$$
$$\mathrm{Pol}_D\, Q := \{f \in \mathrm{O}_D \mid \forall \varrho \in Q : \quad f \rhd \varrho\}$$

closure operators

$$F \mapsto \mathrm{Pol}_D \,\mathrm{Inv}_D\, F$$
$$Q \mapsto \mathrm{Inv}_D \,\mathrm{Pol}_D\, Q$$

Connection to clones

Lemma

$Q \subseteq \mathrm{R}_D \implies \mathrm{Pol}_D\, Q$ *is a clone.*

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Theorem (Bodnarčuk, Kalužnin, Kotov, Romov 69, Geiger 68)

D finite, $F \subseteq \mathrm{O}_D$ a clone $\implies F = \mathrm{Pol}_D\, Q$ for $Q = \mathrm{Inv}_D\, F$.

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Every clone can be described by relations.

Idea

Reduction of complexity by confining the allowed relations

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Clausal relations

From now on

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Definition (Clausal relation)

$p, q \in \mathbb{N}_+$, $\mathbf{a} = (a_1, \dots, a_p) \in D^p$, $\mathbf{b} = (b_1, \dots, b_q) \in D^q$.

$$\mathrm{R}^{\mathbf{a}}_{\mathbf{b}} := \left\{ (\mathbf{x}, \mathbf{y}) \in D^{p+q} \;\middle|\; \bigvee_{i=1}^{p} x_i \geq a_i \vee \bigvee_{j=1}^{q} y_j \leq b_j \right\}.$$

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Special case: binary clausal relation

$p = q = 1$, $\mathbf{a} = (a) \in D^1$, $\mathbf{b} = (b) \in D^1$.

$$\mathrm{R}^{(a)}_{(b)} = \left\{ (x, y) \in D^2 \;\middle|\; x \geq a \vee y \leq b \right\}.$$

C-clones

Definition (C-clone)

= every clone $\mathrm{Pol}_D\, Q$, where Q is a set of clausal relations.

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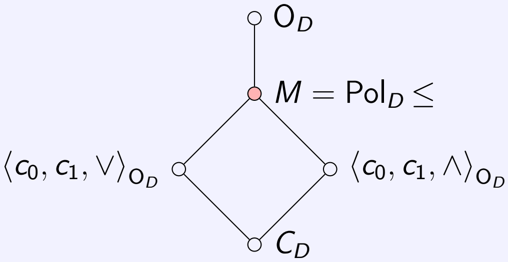
C-clones form a lattice w.r.t. $\subseteq$.

Lattice of C-clones for $D = \{0, 1\}$



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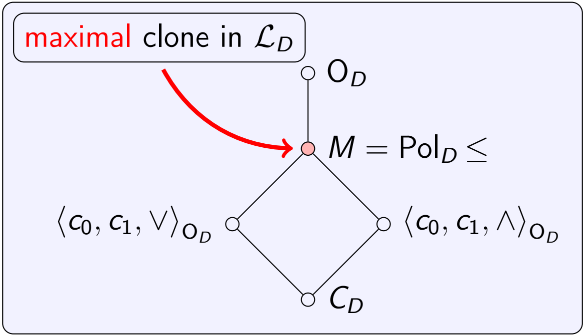

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$$\mathrm{Pol}_D\, \mathrm{R}^{(1)}_{(1)}$$

$$\mathrm{Pol}_D\, \mathrm{R}^{(11)}_{(11)}$$

$$\mathrm{Pol}_D\, \mathrm{R}^{(111)}_{(111)}$$

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- no C-clone = a maximal clone [Beh,Var 2014, submitted]

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- no C-clone = a maximal clone [Beh,Var 2014, submitted]
- $\implies$ every C-clone $\neq \mathrm{O}_D$ satisfies $F \subsetneq M$ for some maximal clone M (as O_D is finitely generated)

The goal

What is the exact relationship?

F .... a maximal C-clone

M .... a maximal clone

$$F \subseteq M$$

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F . . . . a maximal C-clone
M . . . . a maximal clone

$$F \; ? \; M$$

The goal

What is the exact relationship?

F .... a maximal C-clone

M .... a maximal clone

$$F \not\subseteq M$$

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Basics

- maximal clones/C-clones are of the form $\operatorname{Pol}_D \varrho$ for some $\varrho \notin \operatorname{Inv}_D \mathrm{O}_D$.

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- maximal clones/C-clones are of the form $\mathrm{Pol}_D\,\varrho$ for some $\varrho \notin \mathrm{Inv}_D\,\mathrm{O}_D$.
- describing relations are known
- every clone/C-clone $F \subsetneq \mathrm{O}_D$ is a subset of some maximal clone/C-clone. (O_D finitely generated / [Var11])

Maximal C-clones

Theorem (Var11)

A C-clone $F \subseteq \mathrm{O}_D$ is maximal if and only if $F = \mathrm{Pol}_D \left\{ \mathrm{R}^{(a)}_{(b)} \right\}$ for some $a > 0$ and some $b < n - 1$.

Maximal clones

Theorem (I.G. Rosenberg, 1970)

D finite, $F \in \mathcal{L}_D$ maximal iff $F = \mathrm{Pol}_D\{\varrho\}$, for ϱ

1. *partial order with least and greatest element.*
2. *graph $\{(x, f(x)) \mid x \in D\}$ of a prime permutation f.*
3. *non-trivial equivalence relation on D.*
4. *affine relation w.r.t. some elementary Abelian p-group on D, p prime.*
5. *a central relation of arity h $(1 \leq h < |D|)$.*
6. *an h-regular relation $(3 \leq h \leq |D|)$.*

2 Cases

Suppose $a > 0$, $b < n - 1$ and $\varrho \in \mathrm{R}_D$ is a Rosenberg relation.

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$\mathrm{Pol}_D\, \mathrm{R}^{(a)}_{(b)} \subseteq \mathrm{Pol}_D\, \varrho$

$\iff \varrho \in \left[\mathrm{R}^{(a)}_{(b)}\right]_{\mathrm{R}_D}$ find a primitive positive formula for ϱ

O_D

bounded orders	affine rel's	graphs of prime perm	h-regular rel's	central rel's	equiva-lences

$$\mathrm{Pol}_D \left\{ \mathrm{R}^{(a)}_{(b)} \right\} \overset{?}{\subseteq}$$

O_D

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$\varrho = \mathrm{graph}(s),\ s\colon D \longrightarrow D$

$$O_D$$

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- $f \in \mathrm{Pol}_D^{(n)} \varrho \iff s\left(f\left(x_1, \ldots, x_n\right)\right) = f\left(s(x_1), \ldots, s(x_n)\right)$

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- $c_a \in \mathrm{Pol}^{(1)}_D \varrho \iff s(a) = a.$

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- prime permutations have no fixed points

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$\Longrightarrow$ $c_a \in \mathrm{Pol}_D \mathrm{R}^{(a)}_{(b)} \setminus \mathrm{Pol}_D \varrho$

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- central rel's are totally reflexive by definition

$$\mathrm{O}_D$$

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ϱ affine rel w.r.t. GF(p)-vector space structure on D

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- $f \in \mathrm{Pol}^{(n)}_D \varrho \iff f\colon D^n \longrightarrow D$ affine linear function

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- $f \in \mathrm{Pol}_D^{(n)}\,\varrho \iff f\colon D^n \longrightarrow D$ affine linear function

$\Longrightarrow$ $\mathrm{Ker}\,(f - f\,(\mathbf{0},\ldots,\mathbf{0})) = f^{-1}\,[\{f\,(\mathbf{0},\ldots,\mathbf{0})\}] \leq D^n$

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- $\exists f \in \mathrm{Pol}^{(1)}_D\, \mathrm{R}^{(a)}_{(b)}\colon\ \left|f^{-1}\,[\{f\,(\mathbf{0},\dots,\mathbf{0})\}]\right|$ is no power of p.

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- technical case distinction showing

O_D

bounded orders	affine rel's	graphs of prime perm	h-regular rel's	central rel's	equiva-lences
	$\exists f \notin \mathrm{Pol}_D\,\varrho$	$c_a \notin \mathrm{Pol}_D\,\varrho$	$\vee_D \notin \mathrm{Pol}_D\,\varrho$	$\mathrm{ar}(\varrho) \leq 2$	

$$\mathrm{Pol}_D \left\{ \mathrm{R}^{(a)}_{(b)} \right\} \overset{?}{\subseteq}$$

ϱ bounded partial order relation

- technical case distinction showing
- $\exists f \in \mathrm{Pol}^{(\leq 2)}_D\, \mathrm{R}^{(a)}_{(b)} \setminus \mathrm{Pol}_D\,\varrho$ (i.e. f not monotone w.r.t. ϱ)

$$\mathrm{O}_D$$

bounded orders	affine rel's	graphs of prime perm	h-regular rel's	central rel's	equivalences
$\exists f \notin \mathrm{Pol}_D\, \varrho$	$\exists f \notin \mathrm{Pol}_D\, \varrho$	$c_a \notin \mathrm{Pol}_D\, \varrho$	$\vee_D \notin \mathrm{Pol}_D\, \varrho$	$\mathrm{ar}\,(\varrho) \leq 2$	

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O_D

bounded orders	affine rel's	graphs of prime perm	h-regular rel's	central rel's	equiva-lences
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$$\mathrm{Pol}_D \left\{ \mathrm{R}^{(a)}_{(b)} \right\} \overset{?}{\subseteq}$$

Theorem

Binary central relations	Unary central rel's $\emptyset \subsetneq \varrho \subsetneq D$	Equivalence relations θ

Theorem

Binary central relations

Unary central rel's $\emptyset \subsetneq \varrho \subsetneq D$

Equivalence relations θ

$\Delta \subsetneq \varrho \subsetneq D^2$ central, then $\mathrm{Pol}_D \, \mathrm{R}^{(a)}_{(b)} \subseteq \mathrm{Pol}_D \, \varrho \iff$

$a - b < 1$ and $\varrho = \{0, \ldots, b\}^2 \cup \{a, \ldots, n-1\}^2$

Theorem

Binary central relations

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$\varrho = (\downarrow b)^2 \cup (\uparrow a)^2$

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$a - b > 1$ and $\varrho = \{0, \ldots, b\} \cup \{a, \ldots, n-1\}$

$\Delta \subsetneq \theta \subsetneq D^2$ equivalence, then $\mathrm{Pol}_D \, \mathrm{R}^{(a)}_{(b)} \subseteq \mathrm{Pol}_D \, \theta \iff$

$a - b = 1$ and $D/\theta = \{\{0, \ldots, b\}, \{a, \ldots, n-1\}\}$

Theorem

Binary central relations

$a - b < 1$,
$\varrho = (\downarrow b)^2 \cup (\uparrow a)^2$

Unary central rel's $\emptyset \subsetneq \varrho \subsetneq D$

$a - b > 1$,
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Equivalence relations θ

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$a - b < 1$,
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Unary central rel's $\emptyset \subsetneq \varrho \subsetneq D$

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Equivalence relations θ

$a - b = 1$,
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Theorem

For finite D every maximal C-clone is contained in exactly one maximal clone.

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$a - b < 1$,
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Equivalence relations θ

$a - b = 1$,
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Theorem

For finite D every maximal C-clone is contained in exactly one maximal clone.

Details, $D = \{0, \ldots, n-1\}$

$n = 2$ $\quad \mathrm{Pol}_D \mathrm{R}_0^1 = M = \mathrm{Pol}_D \leq_2$.

$n \geq 3$ $\quad a - b < 1 \implies \mathrm{Pol}_D \mathrm{R}_{(b)}^{(a)} \subseteq \mathrm{Pol}_D \left\{(\downarrow b)^2 \cup (\uparrow a)^2\right\}$

$a - b > 1 \implies \mathrm{Pol}_D \mathrm{R}_{(b)}^{(a)} \subseteq \mathrm{Pol}_D \{\downarrow b \cup \uparrow a\}$

$a - b = 1 \implies \mathrm{Pol}_D \mathrm{R}_{(b)}^{(a)} \subseteq \mathrm{Pol}_D \{\theta\}, D/\theta = \{\downarrow b, \uparrow a\}$

Final clause

$$\text{Thank} \;\wedge (\neg\text{me}) \wedge\; \text{for} \;\wedge\; \text{your} \;\wedge (\neg\text{inattention})\,.$$

Completeness criterion

Corollary

Let $D = \{0, \ldots, n-1\}$, $n \geq 3$, F be a clausal clone. If

- $\forall\, 0 \leq b < n-1\, \exists f \in F\colon f \not\triangleright \theta_b$, $D/\theta_b = [0, \ldots, b | b+1, \ldots, n-1]$, and
- $\forall\, 0 < a \leq b < n-1\, \exists f \in F\colon f \not\triangleright (\downarrow b)^2 \cup (\uparrow a)^2$, and
- $\forall\, 0 \leq b \leq n-3\, \forall\, 2 \leq k \leq n-1-b\, \exists f \in F\colon$ $f \not\triangleright \downarrow b \cup \uparrow (b+k)$;

then $F = \mathrm{O}_D$.

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