Non-Idempotent Plonka Functions and Hyperidentities

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Introduction

J. Plonka, *On a method of construction of abstract algebras* , 1966.

Definition

An algebra $\mathfrak{U} = (U; \Sigma)$ is a direct system of algeras $(U_i; \Sigma)$ with l.u.b.-property, where $i \in I$, if the following conditions are valid:
i) $U_i \cap U_j = \emptyset$, for all $i, j \in I, i \neq j$;
ii) $U = \bigcup_{i \in I} U_i$;
iii) On the set of the indexes I exists the relation "$\leq$"such that $(I; \leq)$ is an upper semilattice with the following conditions;
iv) if $i \leq j$, then exists a homomorpism $\varphi_{i,j} : (U_i; \Sigma) \mapsto (U_j; \Sigma)$, such that $\varphi_{i,i} = \varepsilon$ and $\varphi_{i,j} \cdot \varphi_{j,k} = \varphi_{i,k}$, for $i \leq j \leq k$, and ε is an identity mapping;
v) for all $A \in \Sigma$ and all $x_1, \ldots, x_n \in Q$ the following equality is valid:

$$A(x_1, \ldots, x_n) = A(\varphi_{i_1,i_0}(x_1), \ldots, \varphi_{i_n,i_0}(x_n)),$$

where $|A| = n, x_1 \in U_{i_1}, \ldots, x_n \in U_{i_n}, i_1, \ldots, i_n \in I, i_0 = sup\{i_1, \ldots, i_n\}$.

Definition

The function $f : U \times U \mapsto U$ is called a Plonka function for the algebra $\mathfrak{U} = (U; \Sigma)$, if it satisfies the following identities:

1. $f(x, f(y, z)) = f(f(x, y), z)$;
2. $f(x, x) = x$;
3. $f(x, f(y, z)) = f(x, f(z, y))$;
4. $f(F_t(x_1, \dots, x_{n(t)}), y) = F_t(f(x_1, y), \dots f(x_{n(t)}, y))$, for all $F_t \in \Sigma$;
5. $f(y, F_t(x_1, \dots, x_{n(t)})) = f(y, F_t(f(y, x_1), \dots f(y, x_{n(t)})))$, for all $F_t \in \Sigma$;
6. $f(F_t(x_1, \dots, x_{n(t)}), x_i) = F_t(x_1, \dots, x_{n(t)})$, for all $F_t \in \Sigma$ and $i = 1, \dots, n(t)$;
7. $f(y, F_t(y, \dots, y)) = y$, for all $F_t \in \Sigma$.

Theorem (Plonka)

To every Plonka function for the algebra $\mathfrak{U} = (U; \Sigma)$ there corresponds a representation of $\mathfrak{U}$ as a system of algebras $(U_i; \Sigma)$ with l.u.b.-property. Moreover, this correspondence is one-to-one.

Weakly idempotent lattices

Definition

An algebra, $(L; \circ)$, with one binary operation is called a weakly idempotent semilattice, if it satisfies the following identities:
1. $a \circ b = b \circ a$;
2. $a \circ (b \circ c) = (a \circ b) \circ c$;
3. $a \circ (a \circ b) = a \circ b$.

Definition

An algebra $(L; \wedge, \vee)$ is called a weakly idempotent lattice, if its reducts $(L; \wedge)$ and $(L; \vee)$ are both weakly idempotent semilattices and the following identities are also satisfied:

$$a \wedge (b \vee a) = a \wedge a, a \vee (b \wedge a) = a \vee a,$$

$$a \wedge a = a \vee a.$$

First, algebras with the system of mentioned identities, were considered by I. Melnik (1973), J. Plonka (1988), E. Graczynska (1990).

Example

$(Z \setminus \{0\}; \wedge, \vee)$, where $x \wedge y = (|x|, |y|)$ and $x \vee y = [|x|, |y|]$,for which $(|x|, |y|)$ and $[|x|, |y|]$ are the greatest common division (gcd) and the least common multiple (lcm) of $|x|$ and $|y|$, is a weak idempotent lattice, which is not a lattice, since $x \wedge x \neq x$ and $x \vee x \neq x$ for negative x.

Definition

An algebra $(L; \wedge, \vee, *, \triangle)$ with four binary operations is called a weakly idempotent bilattice, if the reducts $(L; \wedge, \vee)$, $(L; *, \triangle)$ are weakly idempotent lattices and the following identity is valid: $a \wedge a = a * a$.

If the reducts $(L; \wedge, \vee)$, $(L; *, \triangle)$ are lattices, then the algebra $(L; \wedge, \vee, *, \triangle)$ is called a bilattice.

Hyperidentities

Let us recall that a hyperidentity is a second-ordered formula of the following type,

$$\forall X_1, \dots, X_m \forall x_1, \dots, x_n (w_1 = w_2),$$

where $X_1, \dots, X_m$ are functional variables, and $x_1, \dots, x_n$ are objective variables in the words (terms) of w_1, w_2. Hyperidentities are usually written without the quantifiers, $w_1 = w_2$. We say, that in the algebra, $(Q; F)$, the hyperidentity, $w_1 = w_2$, is satisfied if this equality is valid, when every objective variable and every functional variable in it is replaced by any element from Q and by any operation of the corresponding arity from F (supposing the possibility of such replacement).

Example

In every weakly idempotent lattice $L = (L; \wedge, \vee)$ the following hyperidentity is valid:

$$X(Y(X(x,y),z),Y(y,z)) = Y(X(x,y),z);$$

and this hyperidentity is called an interlaced hyperidentity.

Definition

A weakly idempotent bilattice (bilattice) is called interlaced, if it satisfies the interlaced hyperidentity.

Definition

A weakly idempotent bilattice (bilattice) $(L; \wedge, \vee, *, \triangle)$ is called distributive, if it satisfies the following hyperidentity:

$$X(x, Y(y,z)) = Y(X(x,y), X(x,z)).$$

For the categorical definition of the hyperidentity, the (bi)homomorphisms between two algebras, $(Q; F)$ and $(Q'; F')$, are defined as the pair, $(\varphi; \tilde{\psi})$, of the mappings:

$$\varphi : Q \to Q', \tilde{\psi} : F \to F', |A| = |\tilde{\psi}A|,$$

with the following condition:

$$\varphi(a_1, \dots, a_n) = (\tilde{\psi}A)(\varphi a_1, \dots, \varphi a_n)$$

for any $A \in F, a_1, \dots, a_n \in Q, |A| = n$.
Algebras with their (bi)homomorphisms,$(\varphi; \tilde{\psi})$, (as morphisms) form a category. The product in this category is called a superproduct of algebras and is denoted by $Q \bowtie Q'$ for the two algebras, Q and Q'.

Example

The superproduct of the two weakly idempotent lattices, $(Q; +, \cdot)$ and $(Q'; +, \cdot)$, is a binary algebra, $(Q \times Q'; (+, +), (\cdot, \cdot), (+, \cdot), (\cdot, +))$, with four binary operations, where the pairs of the operations operate component-wise, i.e.

$$(A, B)((x, y), (u, v)) = (A(x, u), B(y, v)),$$

and $Q \bowtie Q'$ is a weakly idempotent interlaced bilattice.

Theorem

For every weakly idempotent interlaced bilattice $L = (L; \wedge, \vee, *.\triangle)$ *there exists an epimorphis* φ *between* L *and the superproduct of two lattices* L_1 *and* L_2*, such that* $\varphi(x) = \varphi(y) \iff x \wedge x = y \wedge y$*; Hence this epimorphism is an isomorphism on the bilattice of the idempotent elements of the weakly idempotent bilattice.*

In a case of interlaced bilattices we obtain an isomorphism between the bilattice and the superproduct of two lattices.
The similar results for bounded distributive and bounded interlaced bilattices were proven by M. Ginsberg, M. Fitting, A. Romanowska, A. Trakul, A. Avron, B. Mobasher, D. Pigozzi, V. Slutski, H. Voutsadakis, G. Gargov, A. Pynko; For interlaced bilattices without bounds – by Yu. Movsisyan, A. Romanowska, J. Smith.

Non-idempotent Plonka Functions

Definition

The binary function $f : U \times U \to U$ is called a non-idempotent Plonka function for the algebra $\mathfrak{U} = (U; \Sigma)$, if it satisfies the following identities:
1. $f(f(x, y), z) = f(x, f(y, z))$;
2.$f(x, x) = F_t(x, \dots, x)$, for every $F_t \in \Sigma$;
3.$f(x, f(y, z)) = f(x, f(z, y))$;
4.$f(F_t(x_1, \dots, x_{n(t)}), y) = F_t(f(x_1, y), \dots f(x_{n(t)}, y))$, for all $F_t \in \Sigma$;
5.$f(y, F_t(x_1, \dots, x_{n(t)})) = f(y, F_t(f(y, x_1), \dots f(y, x_{n(t)})))$, for all $F_t \in \Sigma$;
6.$f(F_t(x_1, \dots, x_{n(t)}), x_i) = F_t(x_1, \dots, x_{n(t)})$ (for all $1 \le i \le n(t)$) and for all $F_t \in \Sigma$;
7.$f(F_t(x_1, \dots, x_{n(t)}), F_t(x_1, \dots, x_{n(t)})) = F_t(x_1, \dots, x_{n(t)})$, for all $F_t \in \Sigma$;
8. $f(x, f(x, y)) = f(x, y)$.

Definition

An algebra $\mathfrak{U} = (U; \Sigma)$ is called a sum of its pairwise disjoint subalgebras $(U_i; \Sigma)$, where $i \in I$, if the following conditions are valid:
i) $U_i \cap U_j = \emptyset$, for all $i, j \in I, i \neq j$;
ii) $U = \bigcup_{i \in I} U_i$;
iii) On the set of the indexes I exists the relation "$\leq$"such that $(I; \leq)$ is an upper semilattice with the following conditions;
iv) if $i \leq j$, then exists a homomorpism $\varphi_{i,j} : (U_i; \Sigma) \mapsto (U_j; \Sigma)$, such tthat $\varphi_{i,i} = \Delta$ and $\varphi_{i,j} \cdot \varphi_{j,k} = \varphi_{i,k}$, for $i \leq j \leq k$,
v) for all $A \in \Sigma$ and all $x_1, \ldots, x_n \in Q$ the following equality is valid:

$$A(x_1, \ldots, x_n) = A(\varphi_{i_1,i_0}(x_1), \ldots, \varphi_{i_n,i_0}(x_n)),$$

где $|A| = n, x_1 \in U_{i_1}, \ldots, x_n \in U_{i_n}, i_1, \ldots, i_n \in I, i_0 = sup\{i_1, \ldots, i_n\}$.

Theorem

To every non-idempotent Plonka function for the algebra $\mathfrak{U} = (U; \Sigma)$ there corresponds a representation of $\mathfrak{U}$ as a sum of its pairwise disjoint subalgebras.

Weakly idempotent quasilattices

Definition

The binary algebra $\mathfrak{U} = (U; \Sigma)$ is called a weakly idempotent quasilattice, if it satisfies the following hyperidentities:

$$X(x,x) = Y(x,x),$$
$$X(x,y) = X(y,x),$$
$$X(x,X(y,z)) = X(X(x,y),z),$$
$$X(x,X(y,y)) = X(x,y),$$
$$X(Y(X(x,y),z),Y(x,z)) = Y(X(x,y),z).$$

Note that each weakly idempotent semilattice, weakly idempotent lattice and the superproduct of weakly idempotent lattices satisfy the above hyperidentities.

Example

Let $L = (L; \wedge, \vee)$ be a weakly idempotent lattice, then the superproduct

$$L \bowtie L = (L \times L; (\wedge, \wedge), (\vee, \vee), (\wedge, \vee), (\vee, \wedge))$$

satisfies all hyperidentities of the variety of weakly idempotent lattices. The subalgebra

$$(L \times L; (\wedge, \wedge), (\wedge, \vee))$$

also satisfies the hyperidentities of the variety of weakly idempotent lattices, but it does not satisfy the low of weak absorption:
$a \wedge (a \vee b) = a \wedge a, a \vee (a \wedge b) = a \vee a$.

Lemma

Every weakly idempotent quasilattice $(Q; X, Y)$ *with two binary operations is a weakly idempotent lattice or a sum of its pairwise disjoint subalgebras, which are weakly idempotent lattices.*

Theorem

For subdirectly irreducible quasilattice $\mathfrak{U} = (U; \Sigma)$, $|\Sigma| \leq 2$.

Theorem

Each hyperidentity of the variety of weakly idempotent lattices is a consequence of the following hyperidentities:

$$\begin{aligned} X(x,x) &= Y(x,x), \\ X(x,y) &= X(y,x), \\ X(x,X(y,z)) &= X(X(x,y),z), \\ X(x,X(y,y)) &= X(x,y), \\ X(Y(X(x,y),z),Y(x,z)) &= Y(X(x,y),z). \end{aligned}$$