Identities valid in orthomodular lattices

Stephen M. Gagola III

School of Mathematics
University of the Witwatersrand

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Orthomodular lattices

Definition

A *bounded lattice* is an algebraic structure $(L, \vee, \wedge, \mathbf{1}, \mathbf{0})$ where $(L, \vee, \wedge)$ is a lattice and $\mathbf{1}, \mathbf{0} \in L$ are elements such that $x \wedge \mathbf{1} = x$ and $x \vee \mathbf{0} = x$ for any $x \in L$. An *ortholattice* is a bounded lattice, L, that has a one-to-one mapping of L onto itself, $x \longmapsto x'$, called *orthocomplementation*, such that

$$x \vee x' = \mathbf{1}$$

$$x \wedge x' = \mathbf{0}$$

$$x'' = x$$

and

$$x \leq y \;\Rightarrow\; y' \leq x'$$

for any $x, y \in L$ where $\leq$ is the partial order of the lattice.

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Orthomodular lattices

Definition

An ortholattice $(L, \vee, \wedge, ', \mathbf{1}, \mathbf{0})$ is called *orthomodular* if it satisfies the condition

$$x \leq y \implies x \vee (x' \wedge y) = y$$

for any $x, y \in L$.

This condition is equivalent to the equality

$$x \vee (x' \wedge (x \vee y)) = x \vee y.$$

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Orthomodular lattices

Definition

In an ortholattice $(L, \vee, \wedge, ', \mathbf{1}, \mathbf{0})$, two elements x and y are said to *commute* if

$$(x \wedge y) \vee (x \wedge y') \vee (x' \wedge y) \vee (x' \wedge y') = \mathbf{1}.$$

Such an equality is equivalent to

$$(x \wedge y) \vee (x \wedge y') = x.$$

Orthomodular lattices

Remark

It was proven by Beran in 1985 that free orthomodular lattices with two generators, usually denoted by $\mathrm{F}(a,b)$, have exactly 96 elements expressed in terms of a and b. Here $\mathrm{F}(a,b)$ is isomorphic to the direct product $\mathrm{MO}_2 \times\ 2^4$ where 2^4 is the 16-element boolean algebra and MO_2 is the 6-element orthomodular lattice of length two. These 96 expressions in terms of a and b, called the Beran expressions, correspond to the 96 binary operations of orthomodular lattices.

Orthomodular lattices

Theorem (Foulis-Holland)

Let L be an orthomodular lattice. If a, b and c are elements such that one of them commutes with the other two then all six distributive laws involving the three elements a, b and c hold. That is, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for any x, y and z in the sublattice generated by $\{a, b, c\}$.

Orthomodular lattices

Lemma

It was shown that the only operations that are associative in all orthomodular lattices are the six with Beran numbers 1, 2, 22, 39, 92 and 96. Namely, associativity holds only for two binary operations (lattice meet and join), two unary operations (left and right projection) and two nulary operations (constants $\mathbf{0}$ and $\mathbf{1}$).

Orthomodular lattices

Theorem (D'Hooghe and Pykacz)

Let L be an orthomodular lattice with a binary operation $*$ whose Beran number is 12, 18, 28, 34, 44 or 82. If $x, y, z \in L$ such that one of them commutes with the other two then $x * (y * z) = (x * y) * z$.

no.	$a * b$
12	$(a \wedge b) \vee (a \wedge b') \vee (a' \wedge b)$
18	$a \wedge (a' \vee b)$
28	$a \vee (a' \wedge b)$
34	$b \wedge (b' \vee a)$
44	$b \vee (b' \wedge a)$
82	$(a \vee b) \wedge (a \vee b') \wedge (a' \vee b)$

Alternative algebras

Definition

An *alternative algebra* is an algebra which does not need to be associative, only alternative. That is,

$$x * (x * y) = (x * x) * y \tag{L}$$

$$(y * x) * x = y * (x * x) \tag{R}$$

for all x and y in the algebra. These are called the *left and right alternative identities* respectively.

Remark

It can be shown that any algebra which satisfies any two of the three identities (L), (R), and

$$(x * y) * x = x * (y * x) \tag{F}$$

satisfies all three identities and is therefore alternative. Identity (F) is often called the *flexible identity*.

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Orthomodular lattices

Theorem

Let $*$ be one of the 96 binary operations on orthomodular lattices. The operation $*$ satisfies all three of the identities (L), (R) and (F) in all orthomodular lattices if and only if its Beran number is in the set $\{1, 2, 16, 18, 22, 23, 28, 34, 38, 39, 44, 81, 92, 96\}$. All other operations satisfy at most one of the identities (L), (R), (F).

Orthomodular lattices

Nonassociative operations satisfying (L), (R) and (F)

no.	$a * b$
16	$(a \wedge b) \vee (a \wedge b') \vee (a' \wedge b) \vee (a' \wedge b')$
18	$a \wedge (a' \vee b)$
23	$(a' \vee b) \wedge (a \vee (a' \wedge b))$
28	$a \vee (a' \wedge b)$
34	$b \wedge (b' \vee a)$
38	$(a \vee b') \wedge (b \vee (b' \wedge a))$
44	$b \vee (b' \wedge a)$
81	$(a \vee b) \wedge (a \vee b') \wedge (a' \vee b) \wedge (a' \vee b')$

Orthomodular lattices

Lemma (Beran)

Suppose L is an ortholattice with $x, y \in L$. If either $x \leq y$ or $x \leq y'$ then x and y commute.

Lemma (Beran)

If L is an orthomodular lattice with $x, y \in L$ then the following are equivalent:

(i) x and y commute;

(ii) $x \wedge (x' \vee y) = x \wedge y$;

(iii) $x \vee (x' \wedge y) = x \vee y$.

Proposition (Beran)

In any orthomodular lattice, if x commutes with y and z, then x commutes with y', $y \vee z$ and $y \wedge z$, as well as with any (ortho-)lattice polynomial in variables y, z.

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Orthomodular lattices

Theorem

Let L be an orthomodular lattice and let $*$ be an operation with the Beran number 18 (Sasaki projection: $a * b = a \wedge (a' \vee b)$).
Then the following properties hold:

(i) If x and y commute then $x * (y * z) = (x * y) * z$.
(ii) If $y \leq z$ then $(x * y) * z = x * (y * z) = x * y$.
(iii) If $x \leq z$ then $(x * y) * z = x * (y * z) = x * y$.

Orthomodular lattices

Theorem

Let L be an orthomodular lattice and let $*$ be an operation with a Beran number in $\{18, 28\}$. Then

$$(x * y * x) * z = (x * y) * (x * z)$$
$$(z * (x * y)) * x = z * (x * y * x)$$
$$((x * y) * z) * x = (x * y) * (z * x)$$

for any $x, y, z \in L$.

Corollary

Let L be an orthomodular lattice and let $*$ be an operation with a Beran number in $\{34, 44\}$. Then

$$z * (x * y * x) = (z * x) * (y * x)$$
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for any $x, y, z \in L$.

Orthomodular lattices

Theorem

Let L be an orthomodular lattice and let $*$ be an operation with the Beran number in $\{18, 28\}$. If $x, y, z \in L$ such that x and y commute then each of the following expressions has a unique output regardless of the order in which the terms are evaluated:

$$x * y * x * z$$

$$x * y * z * x$$

$$y * x * z * x$$

Corollary

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Orthomodular lattices

Theorem

Let L be an orthomodular lattice and let $*$ be an operation with the Beran number in $\{16, 81\}$. If $x, y, z \in L$ such that x commutes with either y or z then each of the following expressions has a unique output regardless of the order in which the terms are evaluated:

$$x * y * x * z$$

$$x * y * z * x$$

$$z * x * y * x$$

Orthomodular lattices

Theorem

Let L be an orthomodular lattice and let $*$ be the operation with the Beran number 23. If $x, y, z \in L$ such that x and z commute then each of the following expressions has a unique output regardless of the order in which the terms are evaluated:

$$z * x * y * x$$

$$x * y * z * x$$

Corollary

Let L be an orthomodular lattice and let $*$ be the operation with Beran number 38. If $x, y, z \in L$ such that x and z commute then each of the following expressions has a unique output regardless of the order in which the terms are evaluated:

$$x * y * x * z$$

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Orthomodular lattices

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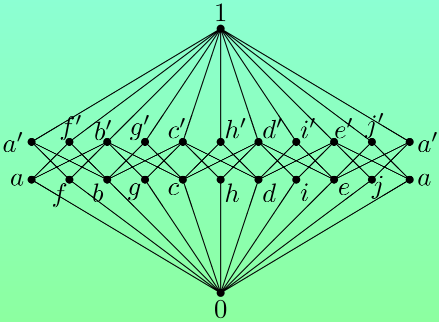
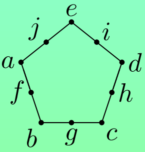
Orthomodular lattices

Remark

It is surprising that all counterexamples can be found in the single orthomodular lattice L_{22}.





The Hasse and Greechie diagrams of an orthomodular lattice L_{22}.

Orthomodular lattices

Question

Does the free orthomodular lattice with three free generators belong to the variety generated by L_{22}?

Questions or comments?