Some properties of strong Galois connections

AAA88

20 - 22 June 2014, Warsaw

Michiro Kondo

Tokyo Denki University, Japan

Let X be a residuated lattice with $f \dashv g$. If $f \dashv g$ satisfies the condition (sGC1): $g(x) \leq x$, then

(sGC2): $g(f(x) \to y) = f(x) \to g(y)$

$$\iff \begin{cases} (1)\ g(g(x)) = g(x) \\ (2)\ g(x) \odot g(y) \leq g(x \odot y) \\ (\mathrm{FS}) : f(x \to y) \leq g(x) \to f(y) \end{cases}$$

Monadic algebras

monadic Heyting algebras (Bezhanisvili 1998)
monadic MV-algebras (DiNola, Grigolia 2004)
monadic BL-algebras (Grigolia 2006)
monadic Rℓ-monoids (Rachůnek, Švrček 2008)
monadic non-com. Rℓ-monoids (Rachůnek, Šalounova 2013)

$$\Downarrow$$

monadic residuated lattices from the view point of ***Galois connection***

$(X, \wedge, \vee, \odot, \rightarrow, 0, 1)$ **is a** *residuated lattice*, **(RL) if**

(1) $(X, \wedge, \vee, 0, 1)$ **is a bounded lattice;**

(2) $(X, \odot, 1)$ **is a commutative monoid;**

(3) For all $x, y, z \in X$,

$$x \odot y \leq z \iff x \leq y \rightarrow z$$

We define $x' = x \rightarrow 0$, **which is a** *negation* **in a sense.**

Proposition 1.

(1) $0' = 1, \;\; 1' = 0$

(2) $x \odot x' = 0$

(3) $x \leq y \iff x \to y = 1$

(4) $x \odot (x \to y) \leq y$

(5) $x \leq y \;\Rightarrow\; x \odot z \leq y \odot z$

(6) $x \leq y \;\Rightarrow\; z \to x \leq z \to y, \;\; y \to z \leq x \to z$

(7) $(x \vee y) \odot z = (x \odot z) \vee (y \odot z)$

(8) $(x \vee y)' = x' \wedge y'$

Let $(P, \leq)$ **be a partially ordered set. A pair of maps** $f, g : P \to P$ **is called a** *Galois connection* **(**$f \dashv g$**) if**

$$f(x) \leq y \Longleftrightarrow x \leq g(y) \quad (x, y \in P)$$

A Galois connection (f, g) **is** *strong* **(**$f \dashv_s g$**) if**

(sGC1) $g(x) \leq x$

(sGC2) $g(f(x) \to y) = f(x) \to g(y)$

(X, f, g) **is a** *residuated lattice with strong Galois connection* **(simply, RLsGC) if** $f \dashv_s g$**.**

Proposition 2. For RLsGC (X, f, g), we have

(1) $f(0) = 0, g(1) = 1$

(2) $x \leq f(x)$

(3) $f(g(x)) = g(x), g(f(x)) = f(x)$

(4) $f(x) = x \Longleftrightarrow g(x) = x$

(5) $f(x \odot y) \leq f(x) \odot f(y),\ g(x) \odot g(y) \leq g(x \odot y)$

(6) $f(f(x) \odot f(y)) = f(x) \odot f(y),$

$g(g(x) \odot g(y)) = g(x) \odot g(y)$

(7) $g(f(x) \to y(y)) = f(x) \to f(y)$

(8) $f(f(x) \wedge f(y)) = f(x) \wedge f(y)$

(9) $g(f(x) \vee f(y)) = f(x) \vee f(y)$

From the above,

$$X_{fg} = \{x \in X \mid f(x) = x\} = \{x \in X \mid g(x) = x\}$$

is a subalgebra of X. A subalgebra X_0 is called *relatively complete* **if, for all $a \in X$, there exist a minimum and a maximum element of the sets $\{x \in X_0 \mid a \leq x\}$ and $\{x \in X_0 \mid x \leq a\}$, respectively. We denote**

$$f_{X_0}(a) = \min\{x \in X_0 \mid a \leq x\}$$
$$g_{X_0}(a) = \max\{x \in X_0 \mid x \leq a\}$$

Lemma 3. For a RLsGC (X, f, g), X_{fg} is a relatively complete subalgebra.

Conversely,

Proposition 4. If X_0 is a relatively complete subalgebra, then

(1) (sGC1) $g_{X_0}(x) \leq x$

(2) $f_{X_0} \dashv g_{X_0}$

(3) (sGC2) $g_{X_0}(f_{X_0}(x) \to y) = f_{X_0}(x) \to g_{X_0}(y)$

From the above,

Lemma 5. **If X_0 is a relatively complete subalgebra, then (X, f_{X_0}, g_{X_0}) is a RLsGC.**

Theorem 6. **Let X be a residuated lattice. Then, there exists a strong Galois connection $f \dashv_s g$ $\Longleftrightarrow$ there exists a relatively complete subalgebra of X.**

$(H, \exists, \forall)$ **is called a** *monadic Heyting algebra* **(mHA) if** H **is a Heyting algebra and** $\exists, \forall$ **satisfy**

(H1) $\forall x \leq x$

(H2) $x \leq \exists x$

(H3) $\forall(x \wedge y) = \forall x \wedge \forall y$

(H4) $\exists(x \vee y) = \exists x \vee \exists y$

(H5) $\forall 1 = 1$

(H6) $\exists 0 = 0$

(H7) $\forall \exists x = \exists x$

(H8) $\exists \forall x = \forall x$

(H9) $\exists(\exists x \wedge y) = \exists x \wedge \exists y$

Proposition 7. For every mHA $(H, \exists, \forall)$, $\exists \dashv_s \forall$.

Proposition 8. $(H, \exists, \forall)$ is an mHA
$\iff$ H is a RLsGC and $\wedge = \odot$.

Corollary 9. For a Heyting algebra H,
$(H, \exists, \forall)$ is an mHA
$\iff$ there exists a relatively complete subalgebra of H.

monadic residuated lattice $(X, \wedge, \vee, \odot, \to, \exists, \forall, 0, 1)$ **is a residuated lattice with** $\exists, \forall$ **satisfying**

(m1) $x \leq \exists x$

(m2) $\forall x \leq x$

(m3) $\forall(x \to \exists y) = \exists x \to \exists y$

(m4) $\forall(\exists x \to y) = \exists x \to \forall y$

(m5) $\forall(x \vee \exists y) = \forall x \vee \exists y$

(m6) $\exists \forall x = \forall x$

(m7) $\forall \forall x = \forall x$

(m8) $\exists(\exists x \odot \exists x) = \exists x \odot \exists x$

(m9) $\exists(x \odot x) = \exists x \odot \exists x$

Basic result: $\exists \dashv_s \forall$**, that is,** $(\exists, \forall)$ **is a strong Galois connection.**

$$\Downarrow$$

Lemma 10. For a residuated lattice X**,**
$(X, \exists, \forall)$ **is a monadic residuated lattice**
$\Longleftrightarrow$ $(X, \exists, \forall)$ **is a RLsGC with (m5) and (m9).**

Application: On Kripke frame (W, R) of modal logic based on CPL,

$$f \dashv g \iff R \text{ is symmetric}$$

$$(\text{sGC1}) : g(x) \leq x \iff R \text{ is reflexive}$$

$$(\text{sGC2}) \iff R \; ?$$

Characterization of (sGC2)

$$\text{(sGC2)} : g(f(x) \to y) = f(x) \to g(y)$$

Lemma 11. Let X be a residuated lattice with $f \dashv g$ and (sGC1): $g(x) \leq x$. The conditions are equivalent:

- **(a) (sGC2):** $g(f(x) \to y) = f(x) \to g(y)$
- **(b)** $g(g(x) \to y) = g(x) \to g(y)$
- **(c)** $f(f(x) \odot y) = f(x) \odot f(y)$
- **(d)** $f(g(x) \odot y) = g(x) \odot f(y)$

Lemma 12. **For any residuated lattice** X **with** $f \dashv g$**, the following conditions are equivalent:**

(i) $f(x) \to g(y) \leq g(x \to y)$

(ii) $f(x) \odot g(y) \leq f(x \odot y)$

(FS): $f(x \to y) \leq g(x) \to f(y)$

Lemma 13. Let X be a residuated lattice with $f \dashv g$ and (sGC1): $g(x) \le x$. Then we have

$$\text{(m3)} : g(x \to f(y)) = f(x) \to g(y)$$

$$\iff \begin{cases} \text{(1)}\ \ g(g(x)) = g(x) \\ \\ \text{(FS)} : f(x \to y) \le g(x) \to f(y) \end{cases}$$

Lemma 14. Let X be a residuated lattice with $f \dashv g$ and (sGC1): $g(x) \leq x$. Then we have

$$\text{(sGC2)} \Longleftrightarrow \begin{cases} \text{(m3)}: \ g(x \to f(y)) = f(x) \to g(y) \\ \\ \text{(2)} \ g(x) \odot g(y) \leq g(x \odot y) \end{cases}$$

Hence we get

Theorem 15. Let X be a residuated lattice with $f \dashv g$ and (sGC1): $g(x) \leq x$.

$$\text{(sGC2)} \Longleftrightarrow \begin{cases} (1)\ g(g(x)) = g(x) \\ (2)\ g(x) \odot g(y) \leq g(x \odot y) \\ (\mathrm{FS}) : f(x \to y) \leq g(x) \to f(y) \end{cases}$$

Proposition 16. **On any Boolean algebra with Galois connection $f \dashv g$,**

$$\textbf{(FS)} \iff f(x) = (g(x'))'$$

This means that

$$\mathsf{CPL} + \{f \dashv g\} + \{gA \to A\} + \{(\mathsf{sGC2})\}$$

$$= \mathsf{CPL} + \{f \dashv g\} + \{gA \to A\}$$
$$+ \{gA \to ggA\} + \{(\mathsf{FS})\}$$

$$= \mathsf{CPL} + \{A \to gfA\} + \{gA \to A\}$$
$$+ \{gA \to ggA\} + \{fA \longleftrightarrow \neg g \neg A\}$$

$$= \mathsf{S5}$$

Hence, in the case of Heyting algebras (intuitionistic logic),

$$\mathsf{Int} + \{f \dashv g\} + \{gA \to A\} + \{(\mathsf{sGC2})\}$$

$$= \mathsf{Int} + \{f \dashv g\} + \{gA \to A\}$$
$$+ \{gA \to ggA\} + \{(\mathsf{FS})\}$$

$$= \mathsf{Int} + \{A \to gfA,\ \ fgA \to A\} + \{gA \to A\}$$
$$+ \{gA \to ggA\} + \{(\mathsf{FS})\}$$

$$= \mathsf{IntS5}\ (= \text{logic of mHA})$$

Thank you for your attention !!