On finite algebras with the basis property

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(with Agnieszka Stocka)

References

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1 General algebras

All algebras considered in this talk, usually A, are finite and have at least one 0-ary operation. If $X \subseteq A$ is a subset then $\langle X \rangle$ is the subalgebra of A generated by X. An element $a \in A$ is a *nongenerator* if it can be rejected from every generating set of A containing this element. Let $\Phi(A)$ denotes the set of all nongenerators of A, (the *Frattini subset of A*).

Proposition 1.1. *Always* $\Phi(A)$ *is the intersection of all maximal subalgebras of* A.

A subset $X \subseteq A$ is said here to be:

- *g-independent* if $\langle Y, \Phi(A)\rangle \neq \langle X, \Phi(A)\rangle$ for every proper subset $Y \subset X$;
- *a g-base of A,* if X is a g-independent generating set of A.

Every algebra has a g-base. Thus we can consider the following g-invariants:

$$\begin{aligned} sg(A) &= \sup\nolimits_X |X| \quad \text{and} \\ ig(A) &= \inf\nolimits_X |X|, \end{aligned} \tag{1}$$

where X runs over all g-bases of A.

Proposition 1.2. *$ig(A) = sg(A) = 0 \Leftrightarrow A$ has no proper subalgebras;*
$ig(A) = sg(A) = 1 \Leftrightarrow A$ has exactly one maximal subalgebra;
In any other case $1 \leq ig(A) \leq sg(A) < \infty$.

Algebras A with $ig(A) = sg(A)$ are named *$\mathcal{B}$-algebras*. An algebra A has *the basis property* if every its subalgebra (in particular A itself) is a $\mathcal{B}$-algebra.

Let $\mathcal{K}$ be a pseudo variety of algebras. Then it is interesting to characterize $\mathcal{B}$-algebras and algebras with the basis property from $\mathcal{K}$.
It is also interesting to connect property $\mathcal{B}$ and the basis property with algebraic operations on algebras from $\mathcal{K}$.

2 Groups

If G is a group then $\Phi(G)$ is a normal subgroup. Hence we can consider the factor group $G/\Phi(G)$.

Theorem 2.1 (Burnside)**.** *Let p be any prime number. If $|G|$ is a power of p (G is a p-group) and $|G/\Phi(G)| = p^r$, then $ig(G) = sg(G) = r$. Hence G has the basis property.*

Example 2.2. Let G be a cyclic group of order $n = p_1^{k_1} \cdot \ldots \cdot p_r^{k_r}$, where p_i are distinct primes and $k_i > 0$ for $i = 1, \ldots, r$. Then $sg(G) = r$, while $ig(G) = 1$. Hence, for $r > 1$, G is not a $\mathcal{B}$-group.

Corollary 2.3. *Let* $1 \leq m \leq n < \infty$*. Then there exists a group* G *such that* $ig(G) = m$ *and* $sg(G) = n$*.*

Theorem 2.4 ([3]). *Let G be a group with the basis property. Then:*

1. *Every element of G has a prime power order;*
2. *G is soluble;*
3. *Every homomorphic image of G has the basis property;*
4. *If $G = G_1 \times G_2$ where G_i are nontrivial, then G has to be a p-group.*

Theorem 2.5 ([1, 5])**.** *Let G be a group. Then G has the basis property if and only if the following conditions are satisfied:*

1. *Every element of G has a prime power order,*
2. *G is a semidirect product of the form $P \rtimes Q$, where P is a p-group and Q is a cyclic q-group, for primes $q \neq p$,*
3. *For every subgroup $H \leq G$, some well defined conditions are satisfied.*

Example 2.6. If P is an elementary abelian 2-group of order 8, Q is a group of order 7 and $G = P \rtimes Q$ is any nonabelian semidirect product of these groups, then G has the basis property. In this group $sg(G) = ig(G) = 2$, but $sg(P) = ig(P) = 3$. Hence, neither ig nor sg is a monotone invariant.

Example 2.7. Consider the group

$$P = \langle a, b \mid a^7 = b^7 = c^7 = 1 = [a,c] = [b,c] \rangle,$$

where $c = [a,b]$. Then $|P| = 7^3$ and every non-trivial element of P has order 7.

Let $Q = \langle x \rangle$ be the group of order 3. Then Q can act on P in the following way:

$$a^{x^j} = a^{2^j} \quad \text{and} \quad b^{x^j} = b^{2^j} \quad \text{for} \quad 1 \leq j \leq 3.$$

Thus, $c^{x^j} = c^{2^{2j}} = c^{4^j}$.

Let $G = P \rtimes Q$ under the above action. Then G is a $\mathcal{B}$-group with elements only of orders $1, 3$ and 7.

If $H = \langle a, c, x \rangle$ then H is not a $\mathcal{B}$-group. Hence G does not satisfy the basis property.

3 Some generalizations

The classes of $\mathcal{B}$-groups and of groups with the basis property are rather narrow. Thus we proposed in [5] a modification of these notions.
A subset $X \subseteq G$ is said there to be:

- *pp-independent* if X is a g-independent set of elements of Prime Power orders;
- a *pp-base* of G, if X is a pp-independent generating set of G.

Then pp-bases exist and the following invariants can be considered:

$$\begin{array}{rcl} s_{pp}(G) & = & \sup_X |X| \quad \text{and} \\ i_{pp}(G) & = & \inf_X |X|, \end{array} \tag{2}$$

where X runs over all pp-bases of G. We also agreed in [5] that a group G is a *$\mathcal{B}_{pp}$-group* if $i_{pp}(G) = s_{pp}(G)$ and G has the *pp-basis property* if all its subgroups are $\mathcal{B}_{pp}$-groups.

The next results are from [5, 6]:

Proposition 3.1. *A group G has the basis property if and only if it has the pp-basis property and every its element is of prime power order.*

Example 3.2. Let

$$G = \langle a, b \mid a^5 = b^4 = 1, a^b = a^4 \rangle.$$

Then G is of order 20 and has the pp-basis property, but it does not have the basis property.

Theorem 3.3. *Let G be a group and $H \leq G$ be a normal subgroup.*

1. *If G is a $\mathcal{B}_{pp}$-group, then G/H is also a $\mathcal{B}_{pp}$-group.*
2. *If G has the pp-basis property, then G/H has also the pp-basis property.*
3. *If G has the pp-basis property, then G is soluble.*

Theorem 3.4. *Let G_1 and G_2 be groups with coprime orders.*

1. *G_1 and G_2 are $\mathcal{B}_{pp}$-groups if and only if $G_1 \times G_2$ is a $\mathcal{B}_{pp}$-group.*
2. *G_1 and G_2 have the pp-basis property if and only if $G_1 \times G_2$ has the pp-basis property.*

Theorem 3.5. *Every nilpotent group has the pp-basis property.*

We proved a structure theorem for groups with pp-basis property. It will be published soon.