Associativity, preassociativity, and string functions

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Associative functions

Let X be a nonempty set.

$F\colon X^2 \to X$ is *associative* if

$$F(F(a,b),c) = F(a,F(b,c)).$$

Examples:

$F(a,b) = a + b$ on $X = \mathbb{R}$

$F(a,b) = a \wedge b$ on a semilattice X

Associative functions

$$F(F(a, b), c) = F(a, F(b, c)).$$

Extension to 3-ary functions

$$F(F(a,b),c) = F(a,F(b,c)).$$

$$F(F(a,b,c),d,e) = F(a,F(b,c,d),e) = F(a,b,F(c,d,e))$$

Associative functions

Extension to *n*-ary functions

$$F(F(a,b),c) = F(a,F(b,c)).$$

$$F(F(a,b,c),d,e) = F(a,F(b,c,d),e) = F(a,b,F(c,d,e))$$

$F\colon X^n \to X$ is *associative* if

$$\begin{aligned}
&F(F(a_1,\dots,a_n),a_{n+1},\dots,a_{2n-1})\\
&= F(a_1,F(a_2,\dots,a_{n+1}),a_{n+2},\dots,a_{2n-1})\\
&= \cdots\\
&= F(a_1,\dots,a_i,F(a_{i+1},\dots,a_{i+n}),a_{i+n+1},\dots,a_{2n-1})\\
&= \cdots\\
&= F(a_1,\dots,a_{n-1},F(a_n,\dots,a_{2n-1})).
\end{aligned}$$

Associative functions with indefinite arity

Let

$$X^* = \bigcup_{n \in \mathbb{N}} X^n.$$

Disclaimer: When we write

$$F \colon X^* \to X,$$

we mean a map

$$F \colon \bigcup_{n \geq 1} X^n \to X$$

that is extended into a map $X^* \to X \cup \{\varepsilon\}$ by setting

$$F(\varepsilon) = \varepsilon.$$

Associative functions with indefinite arity

Let

$$X^* = \bigcup_{n \in \mathbb{N}} X^n.$$

$F\colon X^* \to X$ is *associative* if

$$F(x_1, \dots, x_p, y_1, \dots, y_q, z_1, \dots, z_r) \\ = F(x_1, \dots, x_p, F(y_1, \dots, y_q), z_1, \dots, z_r).$$

Examples: $F(x_1, \dots, x_n) = x_1 + \dots + x_n$ on $X = \mathbb{R}$

$F(x_1, \dots, x_n) = x_1 \wedge \dots \wedge x_n$ on a semilattice X

Notation

We regard n-tuples $\mathbf{x} \in X^n$ as *n-strings* over X.

0-string: ε
1-strings: $x, y, z, \ldots$
n-strings: $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$

X^* is endowed with concatenation.

Example: $\mathbf{x} \in X^n, y \in X, \mathbf{z} \in X^m \implies \mathbf{x}y\mathbf{z} \in X^{n+1+m}$

$$|\mathbf{x}| = \text{length of } \mathbf{x}$$

$$F(\mathbf{x}) = \varepsilon \iff \mathbf{x} = \varepsilon$$

Associative functions with indefinite arity

$F\colon X^* \to X$ is *associative* if

$$F(\mathbf{xyz}) = F(\mathbf{x}F(\mathbf{y})\mathbf{z}) \qquad \forall\, \mathbf{x}, \mathbf{y}, \mathbf{z} \in X^*.$$

Associative functions with indefinite arity

$F\colon X^* \to X$ is *associative* if

$$F(\mathbf{xyz}) = F(\mathbf{x}F(\mathbf{y})\mathbf{z}) \qquad \forall\, \mathbf{x}, \mathbf{y}, \mathbf{z} \in X^*.$$

Equivalent definitions

$$F(F(\mathbf{xy})\mathbf{z}) = F(\mathbf{x}F(\mathbf{yz})) \qquad \forall\, \mathbf{x}, \mathbf{y}, \mathbf{z} \in X^*.$$

$$F(\mathbf{xy}) = F(F(\mathbf{xy})) \qquad \forall\, \mathbf{x}, \mathbf{y} \in X^*.$$

Associative functions with indefinite arity

$F\colon X^* \to X$ is *associative* if

$$F(\mathbf{xyz}) = F(\mathbf{x}F(\mathbf{y})\mathbf{z}) \qquad \forall\ \mathbf{x}, \mathbf{y}, \mathbf{z} \in X^*.$$

Theorem (Marichal, Teheux)

We can assume that $|\mathbf{xz}| \leq 1$ in the definition above.

That is, $F\colon X^* \to X$ is associative if and only if

$$F(\mathbf{y}) = F(F(\mathbf{y}))$$
$$F(x\mathbf{y}) = F(xF(\mathbf{y}))$$
$$F(\mathbf{y}z) = F(F(\mathbf{y})z)$$

Associative functions with indefinite arity

$$F_n = F|_{X^n}$$

$$F_n(x_1 \cdots x_n) = F_2(F_{n-1}(x_1 \cdots x_{n-1})x_n) \qquad n \geq 2$$

Thus, associative functions are completely determined by their unary and binary parts.

Theorem (Marichal)

Let $F\colon X^* \to X$ and $G\colon X^* \to X$ be two associative functions such that $F_1 = G_1$ and $F_2 = G_2$. Then $F = G$.

Preassociative functions

Let Y be a nonempty set.

$F\colon X^* \to Y$ is *preassociative* if

$$F(\mathbf{y}) = F(\mathbf{y}') \quad \Longrightarrow \quad F(\mathbf{x}\mathbf{y}\mathbf{z}) = F(\mathbf{x}\mathbf{y}'\mathbf{z})$$

and

$$F(\mathbf{x}) = F(\varepsilon) \quad \Longleftrightarrow \quad \mathbf{x} = \varepsilon.$$

Preassociative functions

Let Y be a nonempty set.

$F\colon X^* \to Y$ is *string-preassociative* if

$$F(\mathbf{y}) = F(\mathbf{y}') \quad \Longrightarrow \quad F(\mathbf{xyz}) = F(\mathbf{xy}'\mathbf{z}).$$

Preassociative functions

$F\colon X^* \to Y$ is *preassociative* if

$$F(\mathbf{y}) = F(\mathbf{y}') \quad \Longrightarrow \quad F(\mathbf{xyz}) = F(\mathbf{xy'z})$$

and

$$F(\mathbf{x}) = F(\varepsilon) \quad \Longleftrightarrow \quad \mathbf{x} = \varepsilon.$$

Examples: $F(\mathbf{x}) = x_1^2 + \cdots + x_n^2 \quad (X = Y = \mathbb{R})$

$F(\mathbf{x}) = |\mathbf{x}| \quad (X \text{ arbitrary, } Y = \mathbb{N})$

Preassociative functions

$F\colon X^* \to Y$ is *preassociative* if

$$F(\mathbf{y}) = F(\mathbf{y}') \quad \implies \quad F(\mathbf{x}\mathbf{y}\mathbf{z}) = F(\mathbf{x}\mathbf{y}'\mathbf{z})$$

and

$$F(\mathbf{x}) = F(\varepsilon) \quad \iff \quad \mathbf{x} = \varepsilon.$$

Fact: If $F\colon X^* \to X$ is associative, then it is preassociative.

Proof. Suppose $F(\mathbf{y}) = F(\mathbf{y}')$.
Then $F(\mathbf{x}\mathbf{y}\mathbf{z}) = F(\mathbf{x}F(\mathbf{y})\mathbf{z}) = F(\mathbf{x}F(\mathbf{y}')\mathbf{z}) = F(\mathbf{x}\mathbf{y}'\mathbf{z})$.
The second condition holds by definition. □

Preassociative functions

$F\colon X^* \to Y$ is *preassociative* if

$$F(\mathbf{y}) = F(\mathbf{y}') \quad \Longrightarrow \quad F(\mathbf{xyz}) = F(\mathbf{xy}'\mathbf{z})$$

and

$$F(\mathbf{x}) = F(\varepsilon) \quad \Longleftrightarrow \quad \mathbf{x} = \varepsilon.$$

Proposition (Marichal, Teheux)

$F\colon X^* \to X$ is associative if and only if it is preassociative and $F_1(F(\mathbf{x})) = F(\mathbf{x})$.

Proof. (Necessity) Clear.
(Sufficiency) We have $F(\mathbf{y}) = F(F(\mathbf{y}))$. Hence, by preassociativity, $F(\mathbf{xyz}) = F(\mathbf{x}F(\mathbf{y})\mathbf{z})$. □

Preassociative functions

Proposition (Marichal, Teheux)

If $F\colon X^* \to Y$ is preassociative, then so is the function

$$x_1 \cdots x_n \mapsto F_n(g(x_1) \cdots g(x_n))$$

for every function $g\colon X \to X$.

Example: $F_n(\mathbf{x}) = x_1^6 + \cdots + x_n^6 \quad (X = Y = \mathbb{R})$

Preassociative functions

Proposition (Marichal, Teheux)

If $F\colon X^* \to Y$ is preassociative, then so is $g \circ F$ for every function $g\colon Y \to Y$ such that $g|_{\operatorname{Im} F}$ is injective.

Example: $F_n(\mathbf{x}) = \exp(x_1^6 + \cdots + x_n^6) \quad (X = Y = \mathbb{R})$

String functions

A *string function* is a map

$$F\colon X^* \to X^*.$$

$F\colon X^* \to X^*$ is *associative* if

$$F(\mathbf{xyz}) = F(\mathbf{x}F(\mathbf{y})\mathbf{z})$$

and

$$F(\mathbf{x}) = F(\varepsilon) \quad \iff \quad \mathbf{x} = \varepsilon.$$

(the same formula as before!)

String functions

A *string function* is a map

$$F\colon X^* \to X^*.$$

$F\colon X^* \to X^*$ is *string-associative* if

$$F(\mathbf{xyz}) = F(\mathbf{x}F(\mathbf{y})\mathbf{z}).$$

(the same formula as before!)

Associative string functions

Examples:

- identity function
- sorting data in alphabetical order

$$F(\texttt{mathematics}) = \texttt{aacehimmstt}$$
$$F(\texttt{warszawa}) = \texttt{aaarswwz}$$

- removing occurrences of a given letter, say, of $\texttt{a}$

$$F(\texttt{mathematics}) = \texttt{mthemtics}$$
$$F(\texttt{warszawa}) = \texttt{wrszw}$$

- removing duplicates, keeping only the first occurrence of each letter

$$F(\texttt{mathematics}) = \texttt{matheics}$$
$$F(\texttt{warszawa}) = \texttt{warsz}$$

Associative string functions

Examples:

- identity function
- sorting data in alphabetical order

$$F(\texttt{mathematics}) = \texttt{aacehimmstt}$$
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- removing occurrences of a given letter, say, of $\texttt{a}$

$$F(\texttt{mathematics}) = \texttt{mthemtics}$$
$$F(\texttt{warszawa}) = \texttt{wrszw}$$

string-associative
not associative
$F(\texttt{aaa}) = \varepsilon = F(\varepsilon)$

- removing duplicates, keeping only the first occurrence of each letter

$$F(\texttt{mathematics}) = \texttt{matheics}$$
$$F(\texttt{warszawa}) = \texttt{warsz}$$

Associative string functions

$F\colon X^* \to X^*$ is *string-associative* if

$$F(\mathbf{xyz}) = F(\mathbf{x}F(\mathbf{y})\mathbf{z}).$$

Proposition

Assume $F\colon X^* \to X^*$ satisfies $F(\varepsilon) = \varepsilon$. The following are equivalent:

1. *F is string-associative.*
2. *$F(\mathbf{x}F(\mathbf{y})\mathbf{z}) = F(\mathbf{x}'F(\mathbf{y}')\mathbf{z}')$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{x}', \mathbf{y}', \mathbf{z}' \in X^*$ such that $\mathbf{xyz} = \mathbf{x}'\mathbf{y}'\mathbf{z}'$.*
3. *$F(F(\mathbf{xy})\mathbf{z}) = F(\mathbf{x}F(\mathbf{yz}))$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X^*$.*
4. *$F(\mathbf{xy}) = F(F(\mathbf{x})F(\mathbf{y}))$ for all $\mathbf{x}, \mathbf{y} \in X^*$.*

Associative string functions

$F\colon X^* \to X^*$ is *string-associative* if

$$F(\mathbf{xyz}) = F(\mathbf{x}F(\mathbf{y})\mathbf{z}).$$

Proposition

$F\colon X^* \to X^*$ is string-associative if and only if

$$F(\mathbf{xyz}) = F(\mathbf{x}F(\mathbf{y})\mathbf{z})$$

for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X^*$ such that $|\mathbf{xy}| \leq 1$.

Associative string functions

$F\colon X^* \to X^*$ is *string-associative* if

$$F(\mathbf{xyz}) = F(\mathbf{x}F(\mathbf{y})\mathbf{z}).$$

Fact

A string-associative function $F\colon X^* \to X^*$ satisfies

$$F(x_1 \cdots x_n) = F(F(x_1 \cdots x_{n-1})x_n), \quad n \geq 1,$$

or, equivalently,

$$F(x_1 \cdots x_n) = F(F(\cdots F(F(x_1)x_2)\cdots)x_n), \quad n \geq 1.$$

Associative string functions

$F\colon D \to X^*$ is *m-bounded* if $|F(\mathbf{x})| \leq m$ for every $\mathbf{x} \in D$.

Note: Functions $F\colon X^* \to X$ are 1-bounded string functions.

Proposition

Assume that $F\colon X^* \to X^*$ is string-associative, and let $m \in \mathbb{N}$.

1. *F is m-bounded if and only if $F_0, \ldots, F_{m+1}$ are m-bounded.*
2. *If F is m-bounded and $G\colon X^* \to X^*$ is an m-bounded and string-associative function satisfying $G_i = F_i$ for $i = 0, \ldots, m+1$, then $F = G$.*

Associative string functions

$F\colon D \to X^*$ is *m-bounded* if $|F(\mathbf{x})| \leq m$ for every $\mathbf{x} \in D$.

Note: Functions $F\colon X^* \to X$ are 1-bounded string functions.

Proposition

Let $m \in \mathbb{N}$. An m-bounded function $F\colon X^* \to X^*$ is string-associative if and only if

1. $F \circ F_k = F_k$ *for* $k = 0, 1, \ldots, m+1$;
2. $F(F(x\mathbf{y})z) = F(xF(\mathbf{y}z))$ *for all* $x \in X$, $\mathbf{y} \in X^*$, $z \in X$ *such that* $|x\mathbf{y}z| \leq m+2$; *and*
3. $F(x_1 \cdots x_n) = F(F(x_1 \cdots x_{n-1})x_n)$, $n \geq 1$.

Preassociativity and string functions

Theorem

Assume AC. Let $F\colon X^* \to Y$. The following are equivalent.

1. *F is string-preassociative.*
2. *There exists a string-associative string function $H\colon X^* \to X^*$ and an injective function $f\colon \operatorname{Im}(H) \to Y$ such that $F = f \circ H$.*

Preassociativity and string functions

Theorem

Assume AC. Let $F\colon X^* \to Y$*. The following are equivalent.*

1. F *is ~~string~~-preassociative.*
2. *There exists a ~~string~~-associative string function* $H\colon X^* \to X^*$
   and an injective function $f\colon \operatorname{Im}(H) \to Y$
   such that $F = f \circ H$*.*

Theorem

Assume AC. Let $F\colon X^* \to Y$. The following are equivalent.

1. *F is preassociative.*
2. *There exists an associative string function $H\colon X^* \to X^*$ and an injective function $f\colon \operatorname{Im}(H) \to Y$ such that $F = f \circ H$.*

Associative string functions

Proposition

$F\colon X^* \to X^*$ is string-associative and depends only on the length of its input if and only if $F = \psi \circ \alpha \circ |\cdot|$ for some $\psi\colon \mathbb{N} \to X^*$ satisfying $|\psi(n)| = n$ for all $n \in \mathbb{N}$ and $\alpha\colon \mathbb{N} \to \mathbb{N}$ satisfying

$$\alpha(n+k) = \alpha(\alpha(n)+k) \qquad \forall\, n, k \in \mathbb{N}.$$

In this case F is associative if and only if α satisfies

$$\alpha(n) = 0 \quad \iff \quad n = 0.$$

Associative string functions

Proposition

Let $\alpha\colon \mathbb{N} \to \mathbb{N}$. *Then* α *satisfies condition*

$$\alpha(n+k) = \alpha(\alpha(n)+k) \qquad \forall\, n, k \in \mathbb{N}$$

if and only if $\alpha = \mathrm{id}$ *or there exist integers* $n_1 \geq 0$ *and* $\ell > 0$ *such that*

1. $\alpha(n) = n$ *whenever* $0 \leq n < n_1$,
2. α *is ultimately periodic, starting at* n_1, *with period* ℓ,
3. $\alpha(n) \geq n$ *and* $\alpha(n) \equiv n \pmod{\ell}$ *whenever* $n_1 \leq n < n_1 + \ell$.

In addition, α *satisfies condition*

$$\alpha(n) = 0 \quad \Longleftrightarrow \quad n = 0$$

if and only if $\alpha = \mathrm{id}$ *or* α *satisfies conditions 1–3 with* $n_1 > 0$.

Associative string functions

Proposition

Let $\alpha\colon \mathbb{N} \to \mathbb{N}$*. Then* α *satisfies condition*

$$\alpha(n+k) = \alpha(\alpha(n)+k) \qquad \forall\, n,k \in \mathbb{N}$$

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if and only if $\alpha = \mathrm{id}$ *or* α *satisfies conditions 1–3 with* $n_1 > 0$.

Dziękuję.

Kiitos.

Merci.

Obrigado.

Thank you.