Finite groups with some *CEP*-subgroups

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A subgroup H of a group G is an *NR-subgroup* of G (*Normal Restriction*) if, whenever $N \trianglelefteq H$, $N^G \cap H = N$, where N^G is the *normal closure* of N in G (the smallest normal subgroup of G containing N).

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A subgroup H of a group G is *normal sensitive* in G if the following holds:

$$\{N \mid N \text{ is normal in } H\} = \{H \cap L \mid L \text{ is normal in } G\}.$$

A group G is *nilpotent* if it has a *central series*, that is, a normal series $1 = G_0 \leqslant G_1 \leqslant \cdots \leqslant G_n = G$ such that G_{i+1}/G_i is contained in the centre of G/G_i for all i.

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Example

S_3 is a supersoluble group that is not nilpotent.

A_4 is a soluble group that is not supersoluble.

A subgroup H of a group G is a *Hall subgroup* of G if

$$(|H|, |G:H|) = 1.$$

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Every nilpotent group is p-nilpotent;
conversely a group which is p-nilpotent for all p is nilpotent.

Example

$$H = \langle (1\,2)(3\,4) \rangle \lhd V_4 = \langle (1\,2)(3\,4), (1\,3)(2\,4) \rangle \lhd A_4$$

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Let G be a group. A subgroup K of G is *subnormal* in G if there are a non-negative integer r and a series

$$K = K_0 \trianglelefteq K_1 \trianglelefteq K_2 \trianglelefteq \cdots \trianglelefteq K_r = G \quad \text{of subgroups of } G.$$

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Theorem

Let G be a group. Then the following properties are equivalent:

1. *G is nilpotent;*
2. *every subgroup of G is subnormal;*
3. *G is the direct product of its Sylow subgroups.*

A group G is *Dedekind* if every subgroup of G is normal in G.

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Theorem (R. Dedekind, 1896)

A group G is Dedekind if and only if G is abelian or G is a direct product of the quaternion group Q_8 of order 8, an elementary abelian 2-group and an abelian group of odd order.

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Let G be a group. If $N \trianglelefteq G$, then N is permutable in G.

Example

Let p be an odd prime and let G be an extraspecial group of order p^3 and exponent p^2. G has all subgroups permutable, but G has non-normal subgroups.

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Theorem (K. Iwasawa, 1941)

Let p be a prime. A p-group G is an Iwasawa group if and only if G is a Dedekind group, or G contains an abelian normal subgroup N such that G/N is cyclic and so $G = \langle x \rangle N$ for an element x of G and $a^x = a^{1+p^s}$ for all $a \in N$, where $s \geqslant 1$ and $s \geqslant 2$ if $p = 2$.

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Theorem (O.H. Kegel, 1962)

If H is an s-permutable subgroup of G, then H is subnormal in G.

Example

The dihedral group D_8 of order 8 has subgroups which are not permutable but all its subgroups are obviously s-permutable.

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Let G be a group and let α be an automorphism of G. We say that α is a *power automorphism* of G if for every $g \in G$ there exists an integer $n(g)$ such that $n^{\alpha} = g^{n(g)}$. In other words, α is a power automorphism of G if α fixes all the subgroups of G.

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Definition

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Definition

A group G is a *T-group* if every subnormal subgroup of G is normal in G.

Examples of T-groups:

- Dedekind groups = nilpotent T-groups;
- simple groups.

Theorem (W. Gaschütz, 1957)

A group G is a soluble T-group if and only if the following conditions are satisfied:

1. *the nilpotent residual L of G is an abelian Hall subgroup of odd order;*
2. *G acts by conjugation on L as a group of power automorphisms, and*
3. *G/L is a Dedekind group.*

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Definition

A group G is said to be a *PT-group* when
if H is a permutable subgroup of K and K is a permutable subgroup of G, then H is a permutable subgroup of G.

Examples of PT-groups:

- T-groups;
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Examples of PST-groups:

- nilpotent groups;
- PT-groups.

The PST-groups are exactly the groups in which every subnormal subgroup is s-permutable.

Theorem (R.K. Agrawal, 1975)

Let G be a group with nilpotent residual L. The following statements are equivalent:

1. *L is an abelian Hall subgroup of odd order in which G acts by conjugation as a group of power automorphisms;*
2. *G is a soluble PST-group.*

Theorem (R.K. Agrawal, 1975)

Let G be a group with nilpotent residual L. The following statements are equivalent:

1. *L is an abelian Hall subgroup of odd order in which G acts by conjugation as a group of power automorphisms;*
2. *G is a soluble PST-group.*

Corollary

Let G be a group.

1. *G is a soluble PT-group if and only if G is a soluble PST-group whose Sylow subgroups are Iwasawa groups;*
2. *G is a soluble T-group if and only if G is a soluble PST-group whose Sylow subgroups are Dedekind groups.*

Corollary

Every soluble PST-group is supersoluble.

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Example

$S_3 \times S_3$ is a supersoluble group which is not a PST-group.

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The classes of all soluble T-, PT- and PST-groups are closed under taking subgroups.

$$
\begin{array}{ccccc}
T & \subsetneq & PT & \subsetneq & PST \\
\rotatebox{90}{$\subsetneq$} & & \rotatebox{90}{$\subsetneq$} & & \rotatebox{90}{$\subsetneq$} \\
\text{Dedekind} & \subsetneq & \text{Iwasawa} & \subsetneq & \text{nilpotent}
\end{array}
$$

In the soluble universe:

$$
\begin{array}{ccccccc}
T & \subsetneq & PT & \subsetneq & PST & \subsetneq & \text{supersoluble} \\
\rotatebox{90}{$\subsetneq$} & & \rotatebox{90}{$\subsetneq$} & & \rotatebox{90}{$\subsetneq$} & & \\
\text{Dedekind} & \subsetneq & \text{Iwasawa} & \subsetneq & \text{nilpotent} & \subsetneq & \text{supersoluble}
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A group H of a group G is a *CEP-subgroup* of G if whenever N is a normal subgroup of H, there is a normal subgroup L of G such that $N = H \cap L$.

Theorem (S. Bauman, 1974)

Every subgroup of a group G is a CEP-subgroup of G if and only if G is a soluble T-group.

A group H of a group G is a *CEP-subgroup* of G if whenever N is a normal subgroup of H, there is a normal subgroup L of G such that $N = H \cap L$.

Theorem (S. Bauman, 1974)

Every subgroup of a group G is a CEP-subgroup of G if and only if G is a soluble T-group.

Theorem (I.A.M., 2012)

A group G is a soluble T-group if and only if for every $p \in \pi(G)$, every p-subgroup of G is a CEP-subgroup of G.

Let p be a prime. A group G satisfies *the property CEP_p* if a Sylow p-subgroup of G is a CEP-subgroup of G.

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Theorem (I.A.M. 2013)

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A group G is a soluble PST-group if and only if every subgroup of G satisfies CEP_p for all $p \in \pi(G)$.

Theorem (I.A.M. 2014)

Let G be a group. The following conditions are equivalent:

1. *G is a soluble PT-group;*
2. *G satisfies CEP_p and G has Iwasawa Sylow p-subgroups for every $p \in \pi(G)$.*

Theorem (I.A.M., 2013)

If all proper subgroups of even order of a group G satisfy CEP_p for every p, then G is either 2-nilpotent or minimal non-nilpotent. In particular, G is soluble.

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If all proper subgroups of even order of a group G satisfy CEP_p for every p, then G is either 2-nilpotent or minimal non-nilpotent. In particular, G is soluble.

Theorem (S. Li, Y. Zhao, 1988)

Let G be a non-soluble group. Assume that soluble subgroups of G are either 2-nilpotent or minimal non-nilpotent. Then G is one of the following groups:

(1) $PSL(2, 2^f)$, *where f is a positive integer such that $2^f - 1$ is a prime;*

(2) $PSL(2, q)$, *where q is odd, $q > 3$ and $q \equiv 3$ or $5 \pmod 8$;*

(3) $SL(2, q)$, *where q is odd, $q > 3$ and $q \equiv 3$ or $5 \pmod 8$.*

Theorem (I.A.M., 2012)

Let G be a group all of whose second maximal subgroups of even order are soluble PST-groups. Then G is either a soluble group or one of the following groups:

(1) $PSL(2, 2^f)$, *where f is a prime such that $2^f - 1$ is a prime;*

(2) $PSL(2, p)$, *where p is a prime with $p > 3$, $p^2 - 1 \not\equiv 0 \pmod 5$ and $p \equiv 3$ or $5 \pmod 8$;*

(3) $PSL(2, 3^f)$, *where f is an odd prime;*

(4) $SL(2, 3^f)$, *where f is an odd prime and $(3^f - 1)/2$ is a prime;*

(5) $SL(2, p)$, *where p is a prime with $p > 3$, $p^2 - 1 \not\equiv 0 \pmod 5$ and $p \equiv 3$ or $5 \pmod 8$;*

Theorem (I.A.M., 2012)

Let G be a group all of whose second maximal subgroups are soluble PST-groups. Then G is either a soluble group or one of the following groups:

(1) $PSL(2, 2^f)$, *where f is a prime such that $2^f - 1$ is a prime;*

(2) $PSL(2, p)$, *where p is a prime with $p > 3$, $p^2 - 1 \not\equiv 0 \pmod 5$ and $p \equiv 3$ or $5 \pmod 8$;*

(3) $PSL(2, 3^f)$, *where f is an odd prime and $(3^f - 1)/2$ is a prime;*

(4) $SL(2, p)$, *where p is a prime with $p > 3$, $p^2 - 1 \not\equiv 0 \pmod 5$ and $p \equiv 3$ or $5 \pmod 8$.*

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Thank you