Some regular quasivarieties of commutative binary modes

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Commutative binary modes

Binary (or groupoid) modes are algebras with one binary idempotent multiplication satisfying the identity

$$(x \cdot y) \cdot (z \cdot t) = (x \cdot z) \cdot (y \cdot t).$$

Commutative binary modes $(A, \cdot)$ are binary modes with a commutative multiplication, i.e. satisfying the identity

$$x \cdot y = y \cdot x.$$

The class $\mathcal{CBM}_{cl}$ of cancellative commutative binary modes is a subquasivariety of the variety $\mathcal{CBM}$ of commutative binary modes defined by the quasi-identity

$$xy = xz \rightarrow y = z.$$

The class $\mathcal{CBM}_{ir}$ of irregular commutative binary modes is a subquasivariety of the variety $\mathcal{CBM}$ defined by the quasi-identity

$$xy = x \rightarrow x = y.$$

Regularization and Quasi-regularization

The regularization $\widetilde{\mathcal{V}}$ of an irregular idempotent variety $\mathcal{V}$ of groupoids is the smallest variety containing both $\mathcal{V}$ and the variety $\mathcal{S}$ of semilattices.

The quasi-regularization $\widetilde{\mathcal{Q}}^q$ of an irregular idempotent quasivariety $\mathcal{Q}$ of groupoids is the smallest quasivariety containing both $\mathcal{Q}$ and $\mathcal{S}$. If $\mathcal{Q}$ is a variety $\mathcal{V}$, its quasi-regularization $\widetilde{\mathcal{V}}^q$ does not necessary coincide with its regularization $\widetilde{\mathcal{V}}$.

Lemma

For any variety $\mathcal{C}_m = \mathcal{C}_{2k+1} = \mathcal{V}((\mathbb{Z}_{2k+1}, \underline{k+1}))$*, the following conditions are equivalent:*

(a) *The quasi-regularization* $\widetilde{\mathcal{C}}_m^q$ *consists of Płonka sums of groupoids in* $\mathcal{C}_m$ *with injective Płonka homomorphisms;*

(b) $\widetilde{\mathcal{C}}_m^q = \mathsf{SP}(\mathcal{C}_m \cup \mathcal{S})$;

(c) $\widetilde{\mathcal{C}}_m^q$ *is the subquasivariety of* $\widetilde{\mathcal{C}}_m$ *defined by the quasi-identity* (α_m);

(d) $\widetilde{\mathcal{C}}_m^q = \mathsf{P_s}(\{\mathbb{Z}_{p^j} \mid p^j | m\} \cup \{\mathbf{2}_s\})$*, i.e.* $\widetilde{\mathcal{C}}_m^q$ *is generated by all its subdirectly irreducible algebras.*

For a groupoid A and the trivial groupoid $\mathbf{1} = (\{\infty\}, \cdot)$, the symbol A^∞ denotes the Płonka sum of $A_1 = A$ and $A_0 = \{\infty\}$ over the semilattice $\mathbf{2}_s$.

Lemma

Let $m = p_1^{j_1} \dots p_r^{j_r}$, where all p_i are odd prime numbers.

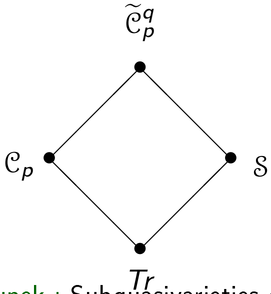
(a) $\widetilde{\mathcal{C}}_m^q = \mathbf{SP}(\mathbb{Z}_{p_1^{j_1}}, \dots, \mathbb{Z}_{p_r^{j_r}}, \mathbf{2}_s)$.

(b) *For $0 < k_i \leq j_i$, the groupoid $\mathbb{Z}_{p_i^{k_i}}^\infty$ does not belong to $\widetilde{\mathcal{C}}_m^q$.*

Theorem

For any variety $\mathcal{C}_m = \mathcal{C}_{2k+1}$, the lattice $\mathcal{L}_q(\widetilde{\mathcal{C}}^q_m)$ of subquasivarieties of the quasi-regularization $\widetilde{\mathcal{C}}^q_m$ is isomorphic to $\mathcal{L}(\mathcal{C}_m) \times \mathbf{2}_l$, the direct product of the lattice of subvarieties of $\mathcal{C}_m$ and the 2-element lattice $\mathbf{2}_l$.

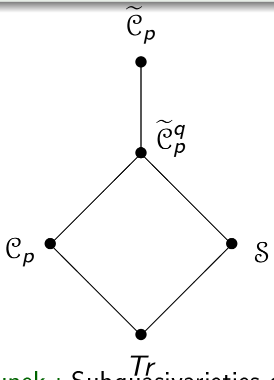
For a prime number p:



Rysunek : Subquasivarieties of $\widetilde{\mathcal{C}}^q_p$

Lemma

Let $p \geq 3$ be a prime number. The lattice $\mathcal{L}_q(\widetilde{\mathcal{C}}_p)$ of subquasivarieties of the regularization $\widetilde{\mathcal{C}}_p$ consists of the five members displayed in Figure:



Rysunek : Subquasivarieties of $\widetilde{\mathcal{C}}_p$

Lemma

Let $m = p_1^{j_1} \dots p_r^{j_r}$*.Then*

$$\widetilde{\mathcal{C}}_m = \mathbf{SP}(\mathbf{Z}^{\infty}_{p_1^{j_1}}, \dots, \mathbf{Z}^{\infty}_{p_r^{j_r}}).$$

Lemma

Let $m = p_1^{j_1} \dots p_r^{j_r}$*. Let* $\mathcal{Q}$ *be a subquasivariety of the regularization* $\widetilde{\mathcal{C}}_m$ *not contained in* $\widetilde{\mathcal{C}}_n$ *for a proper divisor* n *of* m*. Then* $\mathcal{Q}$ *contains* $\widetilde{\mathcal{C}}_m^q$*. Moreover,* $\mathcal{Q}$ *is generated by the subdirectly irreducible* $\mathcal{C}_m$*-groupoids* $\mathbf{Z}_{{p_i}^{j_i}}$ *for* $i = 1, \dots, r$ *and a subset of subdirectly irreducible* $\widetilde{\mathcal{C}}_m$*-groupoids of the form* $\mathbf{Z}^{\infty}_{{p_i}^{k_i}}$*, where* $k_i \leq j_i$*.*

Lemma

Let $p \geq 3$ be a prime number and $j \geq 2$ an integer. Then for $i = 1, \ldots, j$,

$$\widetilde{\mathcal{C}}^q_{p^j} = \mathcal{Q}(\mathbf{Z}_{p^j}, \mathbf{2}_s) = \mathcal{Q}(\mathbf{Z}_{p^j}, \mathbf{Z}^\infty_{p^0}) < \mathcal{Q}(\mathbf{Z}_{p^j}, \mathbf{Z}^\infty_{p^1}) < \ldots$$

$$\cdots < \mathcal{Q}(\mathbf{Z}_{p^j}, \mathbf{Z}^\infty_{p^i}) < \cdots < \mathcal{Q}(\mathbf{Z}_{p^j}, \mathbf{Z}^\infty_{p^j}) = \widetilde{\mathcal{C}}_{p^j}.$$

Each quasivariety $\mathcal{Q}(\mathbf{Z}_{p^j}, \mathbf{Z}^\infty_{p^i})$ is defined by the quasi-identity $(\beta_{p,i})$. Moreover, the quasivarieties $\mathcal{Q}(\mathbf{Z}_{p^j}, \mathbf{Z}^\infty_{p^i})$, for $i = 1, \ldots, j$, form a strictly increasing chain of pairwise distinct subquasivarieties of the regularization $\widetilde{\mathcal{C}}_{p^j}$ properly containing the quasiregularization $\widetilde{\mathcal{C}}^q_{p^j}$.

Let $m = p_1^{j_1} \dots p_r^{j_r}$. Let $\mathcal{Q} = \mathcal{Q}(\mathbf{Z}_m,\ \mathbf{Z}^{\infty}_{{p_1}^{s_1}}, \dots, \mathbf{Z}^{\infty}_{{p_r}^{s_r}})$, where $0 \leq s_i \leq j_i$ for $i \in \{1, \dots, r\}$ be a subquasivariety of $\widetilde{\mathcal{C}}_m$ containing $\widetilde{\mathcal{C}}^q_m$. We will denote:

$$\mathbf{Q}(0, \dots, 0) = \widetilde{\mathcal{C}}^q_m,$$

$$\mathbf{Q}(s_1, \dots, s_r) = \mathcal{Q},$$

and

$$\mathbf{Q}(j_1, \dots, j_r) = \widetilde{\mathcal{C}}_m.$$

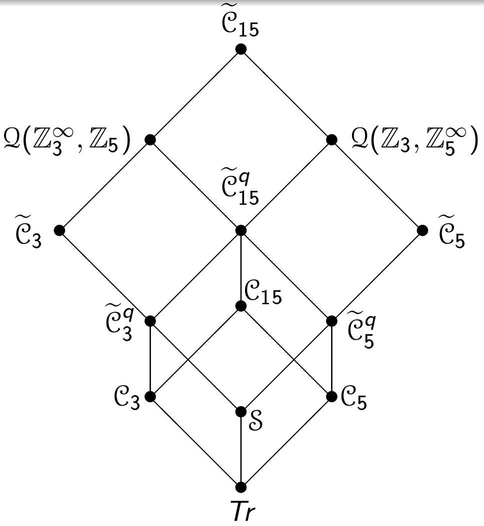
Lemma

Let $m = p_1^{j_1} \dots p_r^{j_r}$. Then for a fixed $i \in \{1, \dots, r\}$, one has

$$\mathbf{Q}(j_1, \dots, j_{i-1}, 0, j_{i+1}, \dots, j_r) < \mathbf{Q}(j_1, \dots, j_{i-1}, 1, j_{i+1}, \dots, j_r) < \dots$$

$$< \mathbf{Q}(j_1, \dots, j_{i-1}, s_i, j_{i+1}, \dots, j_r) < \dots \dots < \mathbf{Q}(j_1, \dots, j_{i-1}, j_i, j_{i+1}, \dots, j_r)$$

Each quasivariety $\mathbf{Q}(j_1, \dots, j_{i-1}, s_i, j_{i+1}, \dots, j_r)$, where $s_i = 0, 1, \dots, j_i$, is defined by (β_{p_i, s_i}). The quasivarieties $\mathbf{Q}(j_1, \dots, j_{i-1}, s_i, j_{i+1}, \dots, j_r)$ form a strictly increasing chain of subquasivarieties of the regularization $\widetilde{\mathcal{C}}_m$ containing the quasivariety $\mathbf{Q}(j_1, \dots, j_{i-1}, 0, j_{i+1}, \dots, j_r)$.



Rysunek : Subquasivarieties of $\widetilde{\mathcal{C}}_{15}$

Let $\mathcal{L}^{>q}(\widetilde{\mathcal{C}}_m)$ denote the lattice of subquasivarieties of $\widetilde{\mathcal{C}}_m$ containing $\widetilde{\mathcal{C}}_m^q$.

Lemma

Let $m = p_1^{j_1} \dots p_r^{j_r}$. Then the lattice $\mathcal{L}^{>q}(\widetilde{\mathcal{C}}_m)$ is isomorphic to the lattice of all divisors of m.

Let us define a certain new lattice $K(m)$. For $m = {p_1}^{m_1} \dots {p_r}^{m_r}$, the set $K(m)$ is the set of functions

$$f : \{0, 1, \dots, 2r\} \to \mathbb{N}$$

satisfying the following conditions:

- $f(0) \in \{0, 1\}$,
- for all $i = 1, \dots, r$, one has $f(i) = j_i$, where $0 \leq j_i \leq m_i$,
- for all $i = r + 1, \dots, 2r$, one has $f(i) = s_i$, where $0 \leq s_i \leq j_i$,
- if $f(0) = 0$, then f for all $i = r + 1, \dots, 2r$, one has $f(i) = 0$.

$K(m)$ is an ordered set with bounds $(0, \dots, 0)$ and $(1, m_1, \dots, m_r, m_1, \dots, m_r)$.

Lemma

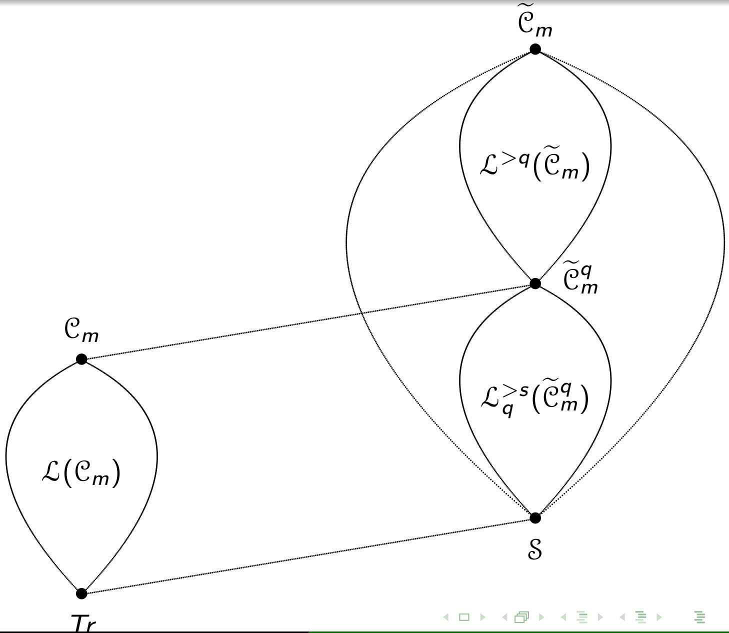
The set $K(m)$ is a distributive lattice with respect to the following operations:

$$(f \vee g)(i) = \max\{f(i), g(i)\}, \ (f \wedge g)(i) = \min\{f(i), g(i)\},$$

where $i \in \{0, 1, \ldots, 2r\}$.

Theorem

The lattice $\mathcal{L}_q(\widetilde{\mathcal{C}}_m)$ of subquasivarieties of the regularization $\widetilde{\mathcal{C}}_m$ is isomorphic to the lattice $K(m)$.


$\widetilde{\mathcal{C}}_m$
$\mathcal{L}^{>q}(\widetilde{\mathcal{C}}_m)$
$\widetilde{\mathcal{C}}_m^q$
$\mathcal{C}_m$
$\mathcal{L}_q^{>s}(\widetilde{\mathcal{C}}_m^q)$
$\mathcal{L}(\mathcal{C}_m)$
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Thank you for your attention.